Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2016

- **5397:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation $\sqrt[3]{x+9} = \sqrt{3} + \sqrt[3]{x-9}$ with $x > 9$.

- **5398:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  If $(2n-1)!! = 1 \cdot 3 \cdot 5 \ldots (2n-1)$, then evaluate

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n}\right).$$

- **5399:** *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

  Let a, b, c be positive real numbers. Prove that

$$\sum_{cyclic} \frac{2a+2b}{\sqrt{6a^2+4ab+6b^2}} \le 3.$$

5400: *Proposed by Arkady Alt, San Jose, CA*

Prove the inequality

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le 12\,(2R-3r),$$

where a, b, c and m_a, m_b, m_c are respectively sides and medians of $\triangle ABC$, with circumradius R and inradius r.

- **5401:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let a, b, c be three positive real numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2} + \frac{c^{-1}}{(4\sqrt{b}+3\sqrt{c})^2} + \frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2} \ge \frac{3}{49}.$$

- **5402:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
$$\int_0^{\infty} \left(\frac{\cos(ax) - \cos(bx)}{x}\right)^2 \mathrm{d}x,$$
  where a and b are real numbers.

Solutions

- **5379:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
$$\frac{(x+1)^4}{(x-1)^2} = 17x.$$

Solution 1 by Ed Gray, Highland Beach, FL

Cross-multiplying and simplifying gives $x^4 - 13x^3 + 40x^2 - 13x + 1 = 0$. Obviously $x \neq 0$, so dividing the polynomial by $40x^2$ gives

$$\begin{aligned} \frac{x^2}{40} - \frac{13}{40}x + 1 - \frac{13}{40} \cdot \frac{1}{x} + \frac{1}{40} \cdot \frac{1}{x^2} &= 0, \\ \frac{1}{40}\left(\left(x^2 + \frac{1}{x^2}\right) - 13\left(x + \frac{1}{x}\right)\right) + 1 &= 0. \end{aligned}$$

Letting $t = x + \dfrac{1}{x}$, squaring $t^2 = x^2 + \dfrac{1}{x^2} + 2$ and then substituting into the above gives

$$\begin{aligned} \frac{1}{40}\left((t^2 - 2) - 13t\right) + 1 &= 0 \\ t^2 - 13t + 38 &= 0, \text{ so} \\ t_1 &= \frac{1}{2}\left(13 + \sqrt{17}\right) \\ t_2 &= \frac{1}{2}\left(13 - \sqrt{17}\right). \end{aligned}$$

Since $t = x + \dfrac{1}{x}$, we have $x^2 - tx + 1 = 0$, and solving for x gives

$$\begin{aligned} x_1 &= \frac{1}{2}\left(t_1 + \sqrt{t_1^2 - 4}\right) \quad x_2 = \frac{1}{2}\left(t_1 - \sqrt{t_1^2 - 4}\right) \\ x_3 &= \frac{1}{2}\left(t_2 + \sqrt{t_2^2 - 4}\right) \quad x_4 = \frac{1}{2}\left(t_2 - \sqrt{t_2^2 - 4}\right). \end{aligned}$$

Substituting in the respective values of t and simplifying gives:

$$x_1 = \frac{13}{4} + \frac{\sqrt{17}}{4} + \frac{1}{2}\sqrt{\frac{85}{2} + \frac{13\sqrt{17}}{2}}$$

$$x_2 = \frac{13}{4}+\frac{\sqrt{17}}{4}-\frac{1}{2}\sqrt{\frac{85}{2}+\frac{13\sqrt{17}}{2}}$$

$$x_3 = \frac{13}{4}-\frac{\sqrt{17}}{4}+\frac{1}{2}\sqrt{\frac{85}{2}-\frac{13\sqrt{17}}{2}}$$

$$x_4 = \frac{13}{4}-\frac{\sqrt{17}}{4}-\frac{1}{2}\sqrt{\frac{85}{2}-\frac{13\sqrt{17}}{2}}.$$

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

Given a real number k, we seek all real solutions of

$$\frac{(x+1)^4}{(x-1)^2} = kx.$$

We require

$$x^4+(4-k)x^3+(6+2k)x^2+(4-k)x+1 = (x^2+ax+1)(x^2+bx+1) = 0,$$

where $a=(4-k+\sqrt{k^2-16k})/2$ and $b=(4-k-\sqrt{k^2-16k})/2$. Hence there are no real solutions unless $k\in(-\infty,0]\cup[16,\infty)$. Solving for x, we obtain $x=(-a\pm\sqrt{a^2-4})/2$ or $x=(-b\pm\sqrt{b^2-4})/2$. We note that if $k=0$, then there is one real solution; if $k<0$ or $k=16$, then there are two real solutions; and if $k>16$, then there are four real solutions.

For the given equation with $k=17$, we have four real solutions:

Letting $a=(-13+\sqrt{17})/2$ and $b=(-13-\sqrt{17})/2$, we obtain

$$x=(-a\pm\sqrt{a^2-4})/2 \approx 4.200 \text{ or } 0.238;$$

$$x=(-b\pm\sqrt{b^2-4})/2 \approx 8.443 \text{ or } 0.118.$$

Comments: **Arkady Alt of San Jose, CA** noted in his solution that the 17 in the statement of the problem could be replaced with any of the three numbers $15, 16$, or 18 to obtain a more elegant answer. For example, the equation

$$\frac{(x+1)^4}{(x-1)^3} = 18x \text{ gives the solutions}$$

$$x = 5\pm 2\sqrt{6} = \left(\sqrt{3}\pm\sqrt{2}\right)^2$$

$$x = 2\pm\sqrt{3} = \left(\frac{\sqrt{6}\pm\sqrt{2}}{2}\right)^2.$$

Kenneth Korbin, proposer of the problem, stated: If $b>2$, then the equation $\frac{(x+1)^4}{(x-1)^3} = \left(4b^2\right)x$ gives the solutions

$$x_1 = \frac{\sqrt{a}+\sqrt{b+\sqrt{c}}}{\sqrt{a}-\sqrt{b+\sqrt{c}}} = \frac{1}{x_2},$$

$$x_3 = \frac{\sqrt{a}+\sqrt{b-\sqrt{c}}}{\sqrt{a}-\sqrt{b-\sqrt{c}}} = \frac{1}{x_4},$$

with $a = 2b$ and with $c = b^2 - 4$.
In the given equation $4b^2 = 17$. Then

$$b^2 = \frac{17}{4}, \quad b = \frac{\sqrt{17}}{2} > 2, \ a = 2b = \sqrt{17}, \ c = b^2 - 4 = \frac{1}{4}.$$

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; Henry Ricardo, New York Math Circle, NY. Toshihiro Shimizu, Kawasaki, Japan; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; (David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5380:** *Proposed by Arkady Alt, San Jose, CA*

Let $\Delta(x, y, z) = 2(xy + yz + xz) - (x^2 + y^2 + z^2)$ and a, b, c be the side-lengths of a triangle ABC. Prove that

$$F^2 \geq \frac{3}{16} \cdot \frac{\Delta(a^3, b^3, c^3)}{\Delta(a, b, c)},$$

where F is the area of $\triangle ABC$.

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

From the Heron's formula,

$$F^2 = \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{16} = \frac{\Delta(a^2, b^2, c^2)}{16}$$

Thus, it suffices to show that $\Delta(a^2, b^2, c^2)\Delta(a, b, c) - 3\Delta(a^3, b^3, c^3) \geq 0 \quad (\heartsuit)$. The (l.h.s) can be written as

$$\sum_{cyc}(a-b)(a-c)q(a, b, c),$$

where $q(a, b, c) = 4a^4 + 2a^3(b + c) + a^2(b - c)^2 \geq 0$. Moreover, since

$$q(a, b, c) - q(b, c, a) = (a - b)(bc^2 + ac^2 + 2b^2c + 2a^2c + 4b^3 + 6ab^2 + 6a^2b + 4a^3),$$

the relation, which is larger $q(a,b,c)$ or $q(b,c,a)$, depends on the value of a or b. Without loss of generality, we assume $a \geq b \geq c$. Then, $q(a,b,c) \geq q(b,c,a) \geq q(c,a,b)$. Thus,

$$\sum_{cyc}(a-b)(a-c)q(a,b,c) = (a-b)\big((a-c)q(a,b,c)-(b-c)q(b,c,a)\big)+q(c,a,b)(a-c)(b-c)$$
$$\geq 0.$$

Therefore, ($\heartsuit$) is true.
Note: It is similar to the proof of Schur's inequality. It seems that ($\heartsuit$) is valid for any a,b,c, even if the constraint that a,b,c are the side-lengths of a triangle is not satisfied.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Denote by s the semiperimeter of the triangle put $s_a = s-a, s_b = s-b, s_c = s-c$. By the triangle inequality, $s_a \geq 0, s_b \geq 0, s_c \geq 0$. Also $a = s_b + s_c,\ b = s_c + s_a,\ c = s_a + s_b$. Furthermore, we note that

$$\Delta(a,b,c) = \Delta\left(s_b+s_c, s_c+s_a, s_a+s_b\right) = 4\left(s_a s_b + s_b s_c + s_c s_a\right) \geq 0.$$

By Heron's formula $F^2 = s \cdot s_a \cdot s_b \cdot s_c = \left(s_a+s_b+s_c\right) s_a \cdot s_b \cdot s_c$.
Therefore we need to prove that

$$64\left(s_a+s_b+s_c\right)\cdot s_a \cdot s_b \cdot s_c\left(s_a s_b + s_b s_c + s_c s_a\right) \geq 3\Delta\left(\left(s_b+s_c\right)^3,\ \left(s_c+s_a\right)^3,\ \left(s_a+s_b\right)^3\right)$$

which is equivalent to

$$27\sum_{symm} s_a^4 s_b^2 + 21\sum_{symm} s_a^3 s_b^3 + 5\sum_{symm} s_a^2 s_b^2 s_c^2 \geq 27\sum_{symm} s_a^4 s_b s_c + 26\sum_{symm} s_a^3 s_b^2 s_c \qquad (1)$$

(as is seen by simply multiplying out).
By Schur's inequality

$$\sum_{cycl} s_a s_b\left(s_a s_b - s_b s_c\right)\left(s_a s_b - s_c s_a\right) \geq 0$$

which is equivalent to

$$\sum_{symm} s_a^3 s_b^3 + \sum_{symm} s_a^2 s_2^2 s_c^2 \geq 2\sum_{symm} s_a^3 s_b^2 s_c \qquad (2)$$

(as is seen again by multiplying out).
We have the following inequalities

$$5\sum_{symm} s_a^3 s_b^3 + 5\sum_{symm} s_a^2 s_b^2 s_c^2 \quad \geq \quad 10\sum_{symm} s_a^3 s_b^2 s_c, \qquad (\text{by}(2)),$$

$$27\sum_{symm} s_a^4 s_b^2 \quad \geq \quad 27\sum_{symm} s_a^4 s_b s_c,\ \text{by Muirhead's inequality},$$

$$16\sum_{symm} s_a^3 s_b^3 \quad \geq \quad 16\sum_{symm} s_a^3 s_b^2 s_c,\ \text{by Muirhead's inequality}.$$

(1) follows by adding these three inequalities.

Solution 3 by proposer

Let $s := \dfrac{t_1+t_2+t_3}{2}$. Since $t_i < s, i = 1,2,3$ (triangle inequalities) then our problem is:
Find $\max s$ for which there are positive integer numbers
t_1, t_2, t_3 satisfying $t_i \le \min\{a_i, s-1\}, i = 1,2,3,\ t_1+t_2+t_3 = 2s$.
First note that $s \ge 3, t_i \ge 2, i = 1,2,3$. Indeed, since $t_i \le s-1$, then
$1 \le s - t_i, i = 1,2,3$ and,
therefore, $t_1 = 2s - t_2 - t_3 = (s-t_2) + (s-t_3) \ge 2$. Cyclic we obtain $t_2, t_3 \ge 2$. Hence,
$2s \ge 6 \iff s \ge 3$.

Since $t_3 = 2s - t_1 - t_2,\ 2 \le t_3 \le \min\{a_3, s-1\}$
then $1 \le 2s - t_1 - t_2 \le \min\{a_3, s-1\} \iff$
$\max\{2s - t_1 - a_3, s+1-t_1\} \le t_2 \le 2s-1-t_1$ and, therefore, for t_2 we obtain the inequality

(1) $\quad \max\{2s - t_1 - a_3, s+1-t_1, 2\} \le t_2 \le \min\{2s-1-t_1, a_2, s-1\}$
with conditions of solvability :

$$\textbf{(2)} \begin{cases} 2s - t_1 - a_3 \le s-1 \\ 2s - t_1 - a_3 \le a_2 \\ s+1-t_1 \le a_2 \\ 2 \le 2s-1-t_1 \end{cases} \iff \begin{cases} s+1-a_3 \le t_1 \\ 2s - a_2 - a_3 \le t_1 \\ s+1-a_2 \le t_1 \\ t_1 \le 2s-3 \end{cases}.$$

Since $s-1 \le 2s-3$ then **(2)** together with $2 \le t_1 \le \min\{a_1, s-1\}$ it gives us bounds for t_1 :

(3) $\max\{s+1-a_3, 2s-a_2-a_3, s+1-a_2, 2\} \le t_1 \le \min\{a_1, s-1\}$.

Since $2 \le a_i, i = 2,3$ then $s+1-a_2 \le s-1, s+1-a_3 \le s-1$ and solvability condition for **(3)** becomes
$s+1-a_3 \le a_1 \iff s \le a_1+a_3-1, 2s-a_2-a_3 \le a_1 \iff s \le \left\lfloor \dfrac{a_1+a_2+a_3}{2} \right\rfloor$,
$s+1-a_2 \le a_1 \iff s \le a_1+a_2-1, 2s-a_2-a_3 \le s-1 \iff s \le a_2+a_3-1$.

Thus, $s^* = \min\left\{ \left\lfloor \dfrac{a_1+a_2+a_3}{2} \right\rfloor, a_1+a_2-1, a_2+a_3-1, a_3+a_1-1 \right\}$ is the largest value of integer semiperimeter.

Solution 4 by Andrea Fanchini, Cantú, Italy

We know that

$$F^2 = s(s-a)(s-b)(s-c)$$

where s is the semiperimeter of $\triangle ABC$.
Now making the substitutions and clearing the denominators we have to prove

$$16s(s-a)(s-b)(s-c)\left[2(ab+bc+ca) - (a^2+b^2+c^2)\right] \ge 3\left[2(a^3b^3+b^3c^3+c^3a^3) - (a^6+b^6+c^6)\right]$$

now we make the following substitutions (with $x, y, z > 0$)

$$a = y+z, \quad b = z+x, \quad c = x+y$$

and expanding out into symmetric sums the given inequality yields
LHS:

$$27(x^4y^2 + x^4z^2 + y^4z^2 + x^2y^4 + x^2z^4 + y^2z^4) + 42(x^3y^3 + y^3z^3 + x^3z^3)+$$

$$+6(x^3yz^2 + x^3y^2z + x^2y^3z + xy^3z^2 + x^2yz^3 + xy^2z^3) + 78x^2y^2z^2$$

RHS:

$$38(x^4yz + xy^4z + xyz^4)$$

so it remains to prove that

$$27(x^4y^2 + x^4z^2 + y^4z^2 + x^2y^4 + x^2z^4 + y^2z^4) \geq 38(x^4yz + xy^4z + xyz^4)$$

or

$$27[4, 2, 0] \geq 19[4, 1, 1]$$

which is true because

$$19[4, 2, 0] \succ 19[4, 1, 1]$$

it follows from Muirhead's Theorem, q.e.d.

Solution 5 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

As well known $F^2 = s(s-a)(s-b)(s-c)$ and $s = (a+b+c)/2$. Upon setting $a = y+z$, $b = x+z$, $c = x+y$,
the inequality becomes

$$\sum_{\text{sym}} \left(27x^4y^2 + 21(xy)^3 + 5(xyz)^2\right) \geq \sum_{\text{sym}} \left(27x^4yz + 26x^3y^2z\right).$$

The third degree Schür inequality is

$$(a^3 + b^3 + c^3) + 3abc \geq \sum_{\text{sym}} a^2b,$$

which applied to the triple $(xy), (yz), (zx)$, yields

$$5\sum_{\text{sym}}(xy)^3 + 5\sum_{\text{sym}}(xyz)^2 \geq 10\sum_{\text{sym}} x^3y^2z.$$

The inequality becomes

$$\sum_{\text{sym}} \left(27x^4y^2 + 16(xy)^3\right) \geq \sum_{\text{sym}} \left(27x^4yz + 16x^3y^2z\right),$$

and the proof is complete upon observing that by the AGM we have

$$x^4y^2 + x^4z^2 \geq 2x^4yz, \qquad (xy)^3 + (xy)^3 + (xz)^3 \geq 3x^3y^2z.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel, and Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

- **5381:** *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

Prove: In any acute triangle ABC, with the usual notations, holds:

$$\sum_{cyclic}\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}\geq\frac{3}{2^{m+1}},$$

where $m\geq 0$ is an integer number.

Solution 1 by Nikos Kalapodis, Patras, Greece

We first recall Barrow's Inequality:
If x,y,z are positive real numbers and $A+B+C=\pi$ then

$$\frac{yz}{2x}+\frac{zx}{2y}+\frac{xy}{2z}\geq x\cos A+y\cos B+z\cos C \qquad (1)$$

(This inequality first appeared in [1]. For a solution see [2] or [3] (inequality 2.20, pp. 23-24)).
Applying inequality (1) for $x=\cos A$, $y=\cos B$, and $z=\cos C$ (note that $\cos A,\cos B,\cos C>0$, since ABC is an acute triangle) we obtain

$$\sum_{cyclic}\frac{\cos A\cos B}{\cos C}\geq 2(\cos^2 A+\cos^2 B+\cos^2 C) \qquad (2)$$

By the following well-known trigonometric identities
$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$ and Euler's inequality ($2r\leq R$) we obtain that $\cos A+\cos B+\cos C=1+\frac{r}{R}\leq\frac{3}{2}$ (3)
Using the AM-GM inequality and inequality (3) we have
$\cos A\cos B\cos C\leq\left(\frac{\cos A+\cos B+\cos C}{3}\right)^3\leq\left(\frac{1}{3}\cdot\frac{3}{2}\right)^3=\frac{1}{8}$ (4)
Furthermore, by the identity $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$ and inequality (4) we obtain

$$\cos^2 A+\cos^2 B+\cos^2 C\geq\frac{3}{4} \qquad (5)$$

By (2) and (5), we have

$$\sum_{cyclic}\frac{\cos A\cos B}{\cos C}\geq\frac{3}{2} \qquad (6)$$

Finally, applying Radon's Inequality and using inequality (6) we have that

$$\sum_{cyclic}\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}=$$

$$\frac{\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}}{1^m}+\frac{\left(\frac{\cos B\cos C}{\cos A}\right)^{m+1}}{1^m}+\frac{\left(\frac{\cos C\cos A}{\cos B}\right)^{m+1}}{1^m}\geq$$

$$\frac{\left(\sum_{cyclic}\frac{\cos A\cos B}{\cos C}\right)^{m+1}}{(1+1+1)^m} \geq \frac{\left(\frac{3}{2}\right)^{m+1}}{3^m} = \frac{3}{2^{m+1}}.$$

References:
[1] L. J. Mordell and D. F. Barrow, Solution 3740, *The American Mathematical Monthly* Vol. 44, No. 4 (Apr., 1937) pp. 252-254
(http://www.jstor.org/stable/2300713)
[2] R. R. Janic, *On A Geometric Inequality Of D. F. Barrow*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-196 (1967), pp.73-74
(http://pefmath2.etf.bg.ac.rs/files/71/194.pdf)
[3] O. Bottema, R. Z. Djordjevic, R. R. Janic, D. S. Mitrinovic, and P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.
Remark: Inequalities (3), (4), and (5) also appear respectively as inequalities 2.16, 2.23 and 2.21 in reference [3].

Solution 2 byÁngel Plaza, University of Las Palmas de Gran Canaria, Spain

By the RMS-AM inequality it is enough to prove that

$$\sum_{cyclic}\frac{\cos A\cos B}{\cos C} \geq \frac{3}{2}.$$

Taking into account that $A+B+C=\pi$, then
$\cos C = \cos\left(\frac{\pi}{2}-A+\frac{\pi}{2}-B\right) = \sin A\sin B - \cos A\cos B$, so the inequality to be proved may be written with cotangents as

$$\sum_{cyclic}\frac{\cot A\cot B}{1-\cot A\cot B} \geq \frac{3}{2}, \text{ or } \sum_{cyclic}\frac{1}{1-\cot A\cot B} \geq \frac{9}{2}.$$

It is well known that if $\alpha=\cot A$, $\beta=\cot B$, and $\gamma=\cot C$, then $\alpha\beta+\beta\gamma+\gamma\alpha=1$. Therefore, taking $x=\alpha\beta$, $y=\beta\gamma$, and $z=\gamma\alpha$ we have to prove that $\sum_{cyclic}\frac{1}{1-x} \geq \frac{9}{2}$ which follows by Jensen's inequality, since function $f(x)=\frac{1}{1-x}$ is convex for $x\in(0,1)$.

Solution 3 by Henry Ricardo, New York Math Circle, NY

Elementary calculations show that for $A, B, C \in (0,\pi/2)$

$$\frac{\cos A\cos B}{\cos C} = \frac{\tan C}{\tan A+\tan B}. \tag{1}$$

Furthermore, we have

$$\sum_{cyclic}\frac{\tan C}{\tan A+\tan B} \geq \frac{3}{2}. \tag{2}$$

by Nesbitt's inequality.

Finally, (1), (2), and the power mean inequality give us

$$\begin{aligned}\sum_{cyclic}\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1} &= \sum_{cyclic}\left(\frac{\tan C}{\tan A+\tan B}\right)^{m+1}\\ &\geq 3\left(\frac{1}{3}\sum_{cyclic}\frac{\tan C}{\tan A+\tan B}\right)^{m+1}\\ &\geq 3\left(\frac{1}{2}\right)^{m+1} = \frac{3}{2^{m+1}}.\end{aligned}$$

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Using the inequality $a^{m+1}+b^{m+1}+c^{m+1}\geq \frac{1}{3^m}(a+b+c)^{m+1}$ (*) we have

$$\sum\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1} = \sum\left(\frac{\tan C}{\tan A+\tan B}\right)^{m+1} \underset{\text{by }(*)}{\geq} \frac{1}{3^m}\left(\sum\frac{\tan C}{\tan A+\tan B}\right)^{m+1}. \quad (**)$$

Setting $\tan A = x,\ \tan B = y,\ \tan C = z$, and using Nesbitt's inequality, we have

$$\sum\frac{\tan C}{\tan A+B} = \sum\frac{z}{x+y} \underset{\text{by Nesbitt)}}{\geq} \frac{3}{2}, \quad (***)$$

The statement of the problem follows from $(**)$ and $(***)$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5382:** *Proposed by* Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Prove that if a, b, c are positive real numbers, then

$$\left(\sum_{cyclic}\frac{a}{b}+8\sum_{cyclic}\frac{b}{a}\right)\left(\sum_{cyclic}\frac{b}{a}+8\sum_{cyclic}\frac{a}{b}\right)\geq 9^3.$$

Solution 1 by Henry Ricardo, New York Math Circle, NY

By the arithmetic-geometric mean inequality, each of the sums $\sum_{cyclic}\frac{a}{b},\ \sum_{cyclic}\frac{b}{a}$ is greater than or equal to 3. Thus

$$\left(\sum_{cyclic}\frac{a}{b}+8\sum_{cyclic}\frac{b}{a}\right)\left(\sum_{cyclic}\frac{b}{a}+8\sum_{cyclic}\frac{a}{b}\right)\geq(3+8\cdot 3)(3+8\cdot 3)=27^2=9^3.$$

Solution 2 by Ed Gray, Highland Beach, FL

Clearly, if $a=b=c$, the above product becomes

$(1+1+1+8(1+1+1))\,(1+1+1+8(1+1+1)) = (3+24)(3+24) = 27^2 = 729 = 9^3.$

Therefore, if we show that the product is minimum when all variables are equal, then the conjecture would be true. It is sufficient to calculate the product in two different ways.

First, suppose that $a = b$ and $c = 0.99a$. Second, suppose $a = b$ and $c = 1.01a$. If both of these products exceed 729, that would show that if all variables are not equal, we do not have a minimum.

Case 1: $a = b, c = 0.99a$. The product becomes

$$\left(1 + \frac{1}{0.99} + 0.99 + 8(1 + 0.99 + \frac{1}{0.99})\right)\left(1 + 0.99 + \frac{1}{0.99} + 8(1 + \frac{1}{0.99} + 0.99)\right) = 729.049$$

Case 2: $a = b, c = 1.01a$. The product becomes

$$\left(1 + \frac{1}{0.99} + 0.99 + 8(1 + 1.01 + \frac{1}{1.01})\right)^2 = 729.048$$

QED

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We begin by applying the extension of the Arithmetic - Geometric Mean Inequality which states that if $\alpha, \beta, x, y > 0$ and $\alpha + \beta = 1$, then

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta},$$

with equality if and only if $x = y$. It follows that

$$\begin{aligned}
\sum_{cyclic} \frac{a}{b} + 8 \sum_{cyclic} \frac{b}{a} &= \sum_{cyclic} \left(\frac{a}{b} + 8\frac{b}{a}\right) \\
&= 9 \sum_{cyclic} \left(\frac{1}{9}\frac{a}{b} + \frac{8}{9}\frac{b}{a}\right) \\
&\geq 9 \sum_{cyclic} \left(\frac{a}{b}\right)^{\frac{1}{9}} \left(\frac{b}{a}\right)^{\frac{8}{9}} \\
&= 9 \sum_{cyclic} \left(\frac{b}{a}\right)^{\frac{7}{9}},
\end{aligned}$$

with equality if and only if $\frac{a}{b} = \frac{b}{a}$, $\frac{b}{c} = \frac{c}{b}$, and $\frac{c}{a} = \frac{a}{c}$, i.e., if and only if $a = b = c$.

Next, apply the standard version of the Arithmetic - Geometric Mean Inequality to get

$$\begin{aligned}
\sum_{cyclic} \frac{a}{b} + 8 \sum_{cyclic} \frac{b}{a} &\geq 9 \sum_{cyclic} \left(\frac{b}{a}\right)^{\frac{7}{9}} \\
&\geq 27 \sqrt[3]{\prod_{cyclic} \left(\frac{b}{a}\right)^{\frac{7}{9}}} \\
&= 27, \qquad (1)
\end{aligned}$$

with equality if and only if $\frac{b}{a} = \frac{c}{b} = \frac{a}{c}$, i.e., if and only if $a = b = c$.

A similar set of steps yields

$$\sum_{cyclic} \frac{b}{a} + 8 \sum_{cyclic} \frac{a}{b} \geq 27, \tag{2}$$

with equality if and only if $a = b = c$.

Therefore, by (1) and (2),

$$\left(\sum_{cyclic} \frac{a}{b} + 8 \sum_{cyclic} \frac{b}{a} \right) \left(\sum_{cyclic} \frac{b}{a} + 8 \sum_{cyclic} \frac{a}{b} \right) \geq 27^2 = 9^3,$$

with equality if and only if $a = b = c$.

Solution 4 by Andrea Fanchini, Cantú, Italy

Clearing the denominators and making the multiplications we have

$$8(a^4b^2 + a^4c^2 + a^2b^4 + b^4c^2 + b^2c^4 + a^2c^4) + 65(a^4bc + ab^4c + abc^4) + 65(a^3b^3 + b^3c^3 + a^3c^3) +$$

$$+16(a^3b^2c + a^3bc^2 + a^2b^3c + ab^3c^2 + a^2bc^3 + ab^2c^3) \geq 534a^2b^2c^2$$

or

$$16[4,2,0] + 65[4,1,1] + 65[3,3,0] + 32[3,2,1] \geq 178[2,2,2]$$

which is true because

$$16[4,2,0] \succ 16[2,2,2]$$

$$65[4,1,1] \succ 65[2,2,2]$$

$$65[3,3,0] \succ 65[2,2,2]$$

$$32[3,2,1] \succ 32[2,2,2]$$

each of which follows from Muirhead's Theorem, q.e.d.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michael Brozinsky (3 solutions), Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel; Nikos Kalapodis, Patras, Greece; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; Neculai Stanciu,"George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Helliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5383: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Find $\gcd(a_n, b_n)$, where a_n and b_n are the positive integers for which $(1 - \sqrt{5})^n = a_n - b_n\sqrt{5}$.

Solution 1 by Ethan Gegner (Undergraduate student, Taylor University), Upland, IN

The equation

$$a_{n+1} - b_{n+1}\sqrt{5} = (a_n - b_n\sqrt{5})(1 - \sqrt{5}) = a_n + 5b_n - (a_n + b_n)\sqrt{5}$$

yields the recurrence relations

$$\begin{aligned} a_{n+1} &= a_n + 5b_n \\ b_{n+1} &= a_n + b_n \end{aligned}$$

Thus,

$$\begin{aligned} \gcd(a_{n+1}, b_{n+1}) = \gcd(a_n + 5b_n, a_n + b_n) &= \gcd(6a_{n-1} + 10b_{n-1}, 2a_{n-1} + 6b_{n-1}) \\ &= \gcd(16a_{n-2} + 40b_{n-2}, 8a_{n-2} + 16b_{n-2}) \\ &= 8\gcd(2a_{n-2} + 5b_{n-2}, a_{n-2} + 2b_{n-2}) \\ &= 8\gcd(b_{n-2}, a_{n-2} + 2b_{n-2}) \\ &= 8\gcd(b_{n-2}, a_{n-2}) \end{aligned}$$

Since $a_1 = b_1 = 1$, $a_2 = 6, b_2 = 2$, $a_3 = 16, b_3 = 8$, we have
$\gcd(a_1, b_1) = 2^0, \gcd(a_2, b_2) = 2^1, \gcd(a_3, b_3) = 2^3$. It follows inductively that

$$\gcd(a_n, b_n) = \begin{cases} 2^n & : 3|n \\ 2^{n-1} & : \text{otherwise} \end{cases}$$

Solution 2 Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

For $n = 1$, $a_1 = b_1 = 1$. For $n > 1$,

$$\begin{aligned} a_n - b_n\sqrt{5} &= \left(a_{n-1} - b_{n-1}\sqrt{5}\right)\left(1 + \sqrt{5}\right) \\ &= a_{n-1} + 5b_{n-1} - \sqrt{5}\,(a_{n-1} + b_{n-1}) \end{aligned}$$

so $a_n = a_{n-1} + 5b_{n-1}$, and $b_n = a_{n-1} + b_{n-1}$, or in matrix form

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \Rightarrow \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, $a_n = 2^{n-1} \cdot L_n$ and $b_n = 2^n \cdot F_n$, where L_n and F_{n+1} respectively are the nth Lucas and the nth Fibonacci numbers. Since $L_n = F_{n-1} + F_{n+1}$, then $gcd(L_n, F_{n+1}) = 1$ and hence $gcd(a_n, b_n) = 2^{n-1}$, if L_n is odd, while $gcd(a_n, b_n) = 2^n$ if L_n is even, that is when n is a multiple of 3.

Solution 3 by Carl Libis, Columbia Southern University, Orange Beach, AL

Let $\left(1 - \sqrt{5}\right)^n = a_n - b_n\sqrt{5}$. Then

$$a_{n+1} - b_{n+1}\sqrt{5} = \left(a_n - b_n\sqrt{5}\right)\left(1 - \sqrt{5}\right) = (a_n + 5b_n) - (a_n + b_n)\sqrt{5}.$$

Thus,

$$(i)\ a_{n+1} = a_n + 5b_n,$$

$$\begin{aligned}
(ii)\ b_{n+1} &= a_n + b_n, \text{ and using (i), and (ii) we can show that}\\
(iii)\ a_{n+1} &= 2a_n + 4a_{n-1},\\
(iv)\ b_{n+1} &= 2b_n + ba_{n-1}. \text{ By observation we note from the first few terms}\\
(v)\ a_n &= 2^{n-1} l_n,\\
(vi)\ b_n &= 2^{n-1} f_n,
\end{aligned}$$

where l_n and f_n are Lucas and Fibonacci numbers. We can verify (v) and (vi) by substituting them into (iii) and (iv).

It is well known that $\gcd(f_n, l_n) = \begin{cases} 2, & \text{if } 3|\ n \\ 1, & \text{otherwise.} \end{cases}$.

See<http://mathhelpforum.com/discrete-math/40492-proof-about-fibonacci-lucas-numbers-gcd.html> or
<https://cms.math.ca/crux/v3/n4/page232-236.pdf>.

Therefore,

$\gcd(a_n, b_n) = \gcd\left(2^{n-1} f_n, 2^{n-1} l_n\right) = 2^{n-1} \begin{cases} 2, & \text{if } 3|\ n \\ 1, & \text{otherwise.} \end{cases} = \begin{cases} 2^n, & \text{if } 3|\ n \\ 2^{n-1}, & \text{otherwise.} \end{cases}$

Comment by Editor: **Kenneth Korbin of NewYork, NY** observed a connection between this problem and the solution to problem 5373, that required us to find positive integers x and y such that $\dfrac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}} - \sqrt{315-135\sqrt{5}}} = \sqrt{x+y\sqrt{5}}$, the unique answer of which was $(x, y) = 161 + 72\sqrt{5}$. He continued on as follows.

Observe that:

$\left(1-\sqrt{5}\right)^{12} = (4096)(161-72\sqrt{5}) = 2^{12}(161-72\sqrt{5})$, and also $(161)^2 - (72\sqrt{5})^2 = 1$. So,

$$\left(161-72\sqrt{5}\right)\left(161+72\sqrt{5}\right)\left(1-\sqrt{5}\right)^{12} = 2^{12}(161-72\sqrt{5})$$

$$161+72\sqrt{5} = \frac{2^{12}}{\left(1-\sqrt{5}\right)^{12}}$$

$$\sqrt{161+72\sqrt{5}} = \frac{2^6}{\left(1-\sqrt{5}\right)^6}.$$

And additionally: $\dfrac{2^6}{\left(1-\sqrt{5}\right)^6} = \dfrac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}} - \sqrt{315-135\sqrt{5}}}$.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China;

Moti Levi, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5384: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : \Re \to \Re$ which verify the functional equation

$$xf'(x) + f(-x) = x^2, \quad \text{for all} \quad x \in \Re.$$

Solution 1 by Michael Brozinsky, Central Islip, NY

We have at once from the given equation $xf'(x) + f(-x) = x^2$ that $f(0) = 0$, and since $x^2 = (-x)^2$ that

$$xf'(x) + f(x) = -xf'(-x) + f(x) \text{ so that}$$
$$x\left(f'(x) + f'(-x)\right) = f(x) - f(-x)$$

which can be cast as $xG'(x) = G(x)$, which we label as equation (1) and in which $G(x) = f(x) - f(-x)$.

From (1) we have $G(x) = cx$ for some constant c and thus $G''(x) = 0$, and so $f''(x) = f''(-x)$ and we label this as equation (2), where we have used the chain rule. Now, by differentiating the given equation twice we have

$$x \cdot f'''(x) + f''(x) + f''(x) + f''(-x) = 2$$

and so from (2) we have

$$f''(0) = \frac{2}{3} \;\; \textit{and} \;\; xf'''(x) + 3f''(x) = 2. \qquad (3)$$

Letting $v = f''(x)$ in (3) we have the linear differential equation $x \cdot \left(\dfrac{dv}{dx}\right) + 3v = 2$, and using the integrating factor x^3 we obtain

$$x^3 dv + 3x^2 v dx \quad = \quad 2x^2 dx \text{ so that}$$

$$x^3 v = \frac{2x^3}{3} + A \quad \textit{and} \quad f''(x) = v = \frac{2}{3} + \frac{A}{x^3} \qquad (4)$$

where the constant$A = 0$ since $f''(0) = \dfrac{2}{3}$. Integrating (4) twice we obtain $f(x) = \dfrac{x^2}{3} + Bx + C$ where B and C are constants and since $f(0) = 0$, we have $C = 0$.

Hence, the general solution is $f(x) = \dfrac{x^2}{3} + Bx$, where B is an *arbitrary* constant.

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

Let $P(x)$ be the given equation. From $P(x) + P(-x)$, we get

$$x\frac{d}{dx}\left(f(x) + f(-x)\right) + \left(f(x) + f(-x)\right) = 2x^2. \qquad (1)$$

From $P(x)-P(-x)$, we get

$$x\frac{d}{dx}\left(f(x)-f(-x)\right)-\left(f(x)-f(-x)\right)=0. \tag{2}$$

First, we solve (1). Let $g(x)=f(x)+f(-x)-\frac{2}{3}x^2$.Then, (1) can be rewritten as

$$x\frac{dg}{dx}=-g(x)$$

The root of this differential equation is $g(x)=C/x$ for constant $C\in R$.
Next, we solve (2). Let $h(x)=f(x)-f(-x)$. Then, (2) can be rewritten as

$$x\frac{dh}{dx}=h(x)$$

The root of this differential equation is $h(x)=Dx$ for constant $D\in R$.
Thus, $f(x)=(g(x)+2/3x^2+h(x))/2=C/x+Dx+x^2/3$ for some constant $C,D\in R$.
Since, $f(x)$ should be defined for all $x\in R$, C must be 0. Therefore, $f(x)=Dx+x^2/3$, where $D\in R$ is a constant and this satisfies $P(x)$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$xf'(x)+f(-x)=x^2,\forall x\in\Re\implies -xf'(-x)+f\left(-(-x)\right)-(-x)^2,\forall x\in\Re$, that is
$-xf'(-x)+f(x)=x^2,\forall x\in\Re\implies xf'(x)+f(-x)=x^2=-xf'\left(-x\right)+f(x),$
$\forall x\in\Re\implies x\left(f'(x)-f'(-x)\right)+f(x)+f(-x)=2x^2,\forall\in\Re$, or equivalently,
$xg'(x)+g(x)=2x^2,\forall x\in\Re$, where $g:\Re\to\Re$ is the function defined by
$g(x)=f(x)+f(-x),\forall x\in\Re$, that is $h'(x)=2x^2$, with $h:\Re\to\Re,\forall x\in\Re$.
$h(x)=xg(x),\forall x\in\Re\implies h(x)=\dfrac{2x^2}{3}+C$, for some $C\in\Re,\forall x\in\Re$
implies $f(x)+f(-x)=g(x)=\dfrac{h(x)}{x}=\dfrac{2x^2}{3}+\dfrac{C}{x},\ \forall x-\{0\}$.
f differentiable implies f differentiable at $x=0\implies f$ continuous at $x=0$.
This fact and the equality $f(x)+f(-x)=\dfrac{2x^2}{3}+\dfrac{C}{x}$ imply that $C=0$.
Hence, $f(-x)=\dfrac{2x^2}{3}-f(x)$ and thus $xf'(x)+\dfrac{2x^2}{3}-f(x)=xf'(x)+f(-x)=x^2$.
$\forall x\in\Re-\{0\}\implies xf'(x)-f(x)=\dfrac{x^2}{3}\forall x\in\Re-0\}\implies\dfrac{f'(x)}{x}-\dfrac{f(x)}{x^2}=\dfrac{1}{3},\forall x\in\Re-\{0\}$.
$\implies k'(x)\dfrac{1}{3}$, where $k:\Re\to\Re$ is the function defined by $k(x)=\dfrac{f(x)}{x},\forall x-\{0\}$
$\implies k(x)=\dfrac{x}{3}+D$ with $D\in\Re,\forall x\in\Re-\{0\}\implies f(x)=\dfrac{x^2}{3}+Dx,\forall x\in\Re-\{0\}$. Since
$\left(f'(x)-f'^{(-x)}\right)+f(x)+f(-x)=2x^2,\forall x\in\Re\implies$
$2f(0)=0\left(f'(0)-f'(-0)\right)+f(0)+f(-0)=2\cdot 0^2=0$, so $f(0)=0$, we conclude that
$f(x)=\dfrac{x^2}{3}+Dx,\forall x$, where D is any real constant.

Solution 4 by Moti Levy, Rehovot, Israel

The derivative of $f:R\to R$ satisfies the functional equation

$$f^{'}(x)=\frac{x^2-f(-x)}{x}, \tag{1}$$

hence it is also differentiable function (maybe except for $x=0$).

Differentiation of the functional equation gives,

$$xf^{''}(x)+f^{'}(x)-f^{'}(-x)=2x. \tag{2}$$

Substitution of (1) into (2) gives,

$$xf^{''}(x)+f^{'}(x)+\frac{x^2-f(x)}{x}=2x,$$

or

$$x^2f^{''}(x)+xf^{'}(x)-f(x)=x^2. \tag{3}$$

All the differentiable functions which satisfy the functional equation $xf^{'}(x)+f(-x)=x^2$, must satisfy (3).
The solutions of the differential equation (3) are

$$f(x)=\frac{1}{3}x^2+\alpha\left(x+\frac{1}{x}\right)+\beta\left(x-\frac{1}{x}\right) \tag{4}$$

Now we substitute (4) in the left side of the original functional equation:

$$x\frac{d\left(\frac{1}{3}x^2+\alpha\left(x+\frac{1}{x}\right)+\beta\left(x-\frac{1}{x}\right)\right)}{dx}+\frac{1}{3}x^2-\alpha\left(x+\frac{1}{x}\right)+\beta\left(\frac{1}{x}-x\right)$$
$$=\frac{1}{x}\left(x^3-2\alpha+2\beta\right)=x^2-\frac{2}{x}(\alpha-\beta).$$

It follows that α must be equal to β for (4) to be a solution.
All the differentiable functions $f:R\to R$, which satisfy the functional equation $xf^{'}(x)+f(-x)=x^2$, for all $x\in R$ are

$$f(x)=\frac{1}{3}x^2+cx, \quad c\in R.$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

Denote the given functional equation by (1). We show that

$$f(x)=\frac{x^2}{3}+kx, \tag{2}$$

where k is an arbitrary constant.

Replacing x bt $-x$ in (1), we obtain

$$-xf'(-x)+f(x)=x^2. \tag{3}$$

Subtracting (3) fromf (1), we obtain

$$x\left(f'(x)+f'(-x)\right)-\left(f(x)-f(-x)\right)=0. \tag{4}$$

Integrating (4), we obtain $f(x) - f(-x) = ax$, where a is an arbitrary constant. By substituting $f(-x) = f(x) - ax$ back into (1). we obtain

$$xf'(x) + f(x) = x^2 + ax. \qquad (5)$$

Integrating (5), we obtain $xf(x) = \dfrac{x^3}{3} + \dfrac{ax^2}{2} + b$, where b is a constant. By putting $x = 0$ we see that $b = 0$. Thus (2) hold for $x \neq 0$. By putting $x = 0$ into (1), we obtain $f(0) = 0$ and so (2) hold for $x = 0$ as well.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Henry Ricardo, New York Math Circle, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

Editor's Notes

The conjecture in **5375*** has been revised by its author **Kenneth Korbin of NY, NY** to the following:

5375* (revised): Prove or disprove the following conjecture. Let k be the product of N different prime numbers each congruent to 1(mod 4).

The total number of different rectangles and trapezoids with integer length sides and diagonals that can be inscribed in a circle with diameter k is exactly $\dfrac{5^N - 3^N}{2}$.

Toshihiro Shimizu of Kawasaki, Japan provided a counter example to the original statement of the problem that did not require the diagonals to also be integers. He let $k = 5 \cdot 17 = 85$ and then developed the trapezoids $(34, 43, 34, 83)$ and $(50, 43, 50, 83)$. The diagonals of these two trapezoids are not of integral length. Ken commented on Toshihiro's examples by saying that: "It never occurred to me that a trapezoid with integer length sides inscribed in a circle with diameter k could have non-integer length diagonals." So with the revision, 5375* remains an open problem.