Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2017

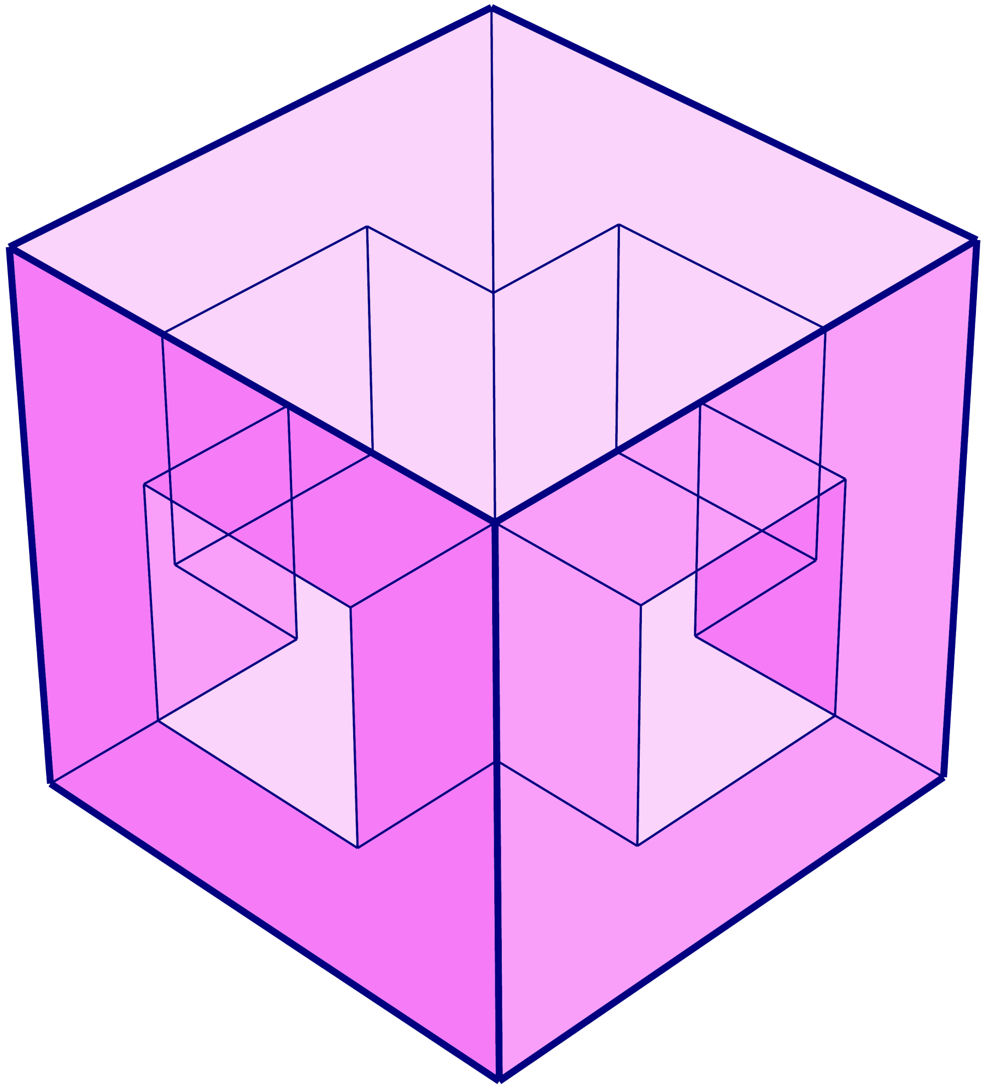
- **5415:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with inradius r and with cevian $\overline{CD}$. Triangle ACD has inradius x and triangle BCD has inradius y, where x, y and r are positive integers with $(x, y, r) = 1$.
  Part 1: Find x, y, and r if $x + y - r = 100$

  Part 2: Find x, y, and r if $x + y - r = 101$.

- **5416:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Two congruent intersecting holes, each with a square cross-section were drilled through a cube. Each of the holes goes through the opposite faces of the cube. Moreover, the edges of each hole are parallel to the appropriate edges of the original cube, and the center of each hole is at the center of the original cube. Letting the length of the original cube be a, find the length of the square cross-section of each hole that will yield the largest surface area of the solid with two intersecting holes. What is the largest surface area of the solid with two intersecting holes?



- **5417:** *Proposed by Arkady Alt, San Jose, CA*

  Prove that for any positive real number x, and for any natural number $n \geq 2$,

$$\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\cdots+x^{n-1}}{n}}.$$

- **5418:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

  Let ABC be an acute triangle with circumradius R and inradius r. If $m \geq 0$, then prove that
  $$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

- **5419:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that
  $$\prod_{k=1}^{n} \left(\sum_{k=1}^{n} a_k^{t_k} \right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{t_{n+1}}{4}} \right)^n$$
  where for all $k \geq 1,\ t_k$ is the k^{th} tetrahedral number defined by $t_k = \dfrac{k(k+1)(k+2)}{6}$.

- **5420:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$. Calculate
  $$\lim_{n\to\infty} \frac{1}{n} \left(I_2 + \frac{A^n}{n} \right)^n.$$

Solutions

- **5397:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation $\sqrt[3]{x+9} = \sqrt{3} + \sqrt[3]{x-9}$ with $x > 9$.

  Solution 1 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

  Cube both sides of the given equation and rearrange to obtain
  $$(x-9)^{2/3} + \sqrt{3}\,(x-9)^{1/3} + \left(1 - 2\sqrt{3}\right) = 0.$$
  This is a quadratic equation with respect to $u = \sqrt[3]{x-9}$ with solutions
  $$u = \frac{-\sqrt{3} \pm \sqrt{8\sqrt{3}-1}}{2}.$$
  When $x > 9$, we have $u > 0$ and
  $$x = 9 + \left(\frac{-\sqrt{3} + \sqrt{8\sqrt{3}-1}}{2} \right)^3$$

$$= \left(1+\sqrt{3}\right)\left(8\sqrt{3}-1\right)^{1/2}$$
$$= \sqrt{44+30\sqrt{3}}.$$

Solution 2 by Brain D. Beasely, Presbyterian College, Clinton, SC

Rewriting the given equation and cubing both sides yields
$$(x+9) - 3\sqrt[3]{(x+9)^2(x-9)} + 3\sqrt[3]{(x+9)(x-9)^2} - (x-9) = 3\sqrt{3},$$
or $3\sqrt[3]{x^2-81}(\sqrt[3]{x-9} - \sqrt[3]{x+9}) = 3\sqrt{3} - 18$. Then $-3\sqrt{3}\sqrt[3]{x^2-81} = 3\sqrt{3}-18$, so cubing once more produces
$$-81\sqrt{3}(x^2-81) = 2997\sqrt{3} - 7290.$$
Hence $x^2 = 30\sqrt{3}+44$, so requiring $x > 9$ yields $x = \sqrt{30\sqrt{3}+44} \approx 9.795995$.

Solution 3 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well known that if $a+b+c=0$, then $a^3+b^3+c^3 = 3abc$. (1)

From the equation we have $\sqrt[3]{x+9} - \sqrt{3} - \sqrt[3]{x-9} = 0$, and with the help of (1) we get

$$x+9-3\sqrt{3}-(x-9) = 3\sqrt{3}\cdot\sqrt[3]{x^2-81}$$
$$18-3\sqrt{3} = 3\sqrt{3}\cdot\sqrt[3]{x^-81},$$ and dividing both sides by $3\sqrt{3}$, gives
$$2\sqrt{3}-1 = \sqrt[3]{x^2-81}. \qquad (2)$$

From (2) we have $(2\sqrt{3}-1)^3 = x^2-81$, which yields $x = \pm\sqrt{81+(2\sqrt{3}-1)^3}$ and since $x > 9$, the only solution is $x = \sqrt{81+(2\sqrt{3}-1)^3} = \sqrt{30\sqrt{3}+44}$.

Solution 4 by Kee-Wai Lau, Hong Kong, China

By the substitution $x = y^3+9$, we obtain $\sqrt[3]{y^3+18} = \sqrt{3}+y$. Cubing both sides and simplifying, we have $y^2+\sqrt{3}y+1-2\sqrt{3}=0$, so that the only positive solution is
$y = \dfrac{\sqrt{8\sqrt{3}-1}-\sqrt{3}}{2}$. Hence the solution to the equation of the problem is
$x = (1+\sqrt{3})\left(\sqrt{8\sqrt{3}-1}\right) = 9.79\cdots$.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Karl Havlak, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, State University of New York at Oneonta, Oneonta, NY; Boris Rays, Brooklyn, NY; Toshihiro Shimizu, Kawaskaki, Japan; Albert, Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota "Traian Vuia Technical College, Focsani, Romania and the proposer.

Students from Taylor University in Upland, IN.
Group 1: Ben Byrd, Maddi Guillaume, and Makayla Schultz (jointly)
Group 2: Rebekah Couch, Hannah Keyser, and Nolan Willoughby (jointly)
Group 3: Michelle Franch, Caleb Knuth, and Savannah Porter (jointly)
Group 4: Lauren Moreland, Anna Souzis, and Boni Hernandez (jointly).

- **5398:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $(2n-1)!! = 1\cdot 3\cdot 5\ldots(2n-1)$, then evaluate

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{n!(2n-1)!!}}{n}\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

By Stirling's asymptotic formula,

$$n! = \left(\sqrt{2\pi n}\right) n^n e^{-n+O\left(\frac{1}{n}\right)}, \text{ as } n\to\infty.$$

So

$$\begin{aligned}\frac{\sqrt[n]{n!\,(2n-1)!!}}{n} = \frac{1}{n}\sqrt[n]{\frac{(2n)!}{2^n n!}} = \frac{1}{2n}\sqrt[n]{(2n)!} &= \frac{1}{2n}\sqrt[2n]{4\pi n}\,(2n)^2\,e^{-2+O\left(\frac{1}{n^2}\right)}\\ &= \frac{2n}{e^2}e^{\frac{\ln(4\pi n)}{2n}+O\left(\frac{1}{n^2}\right)}\\ &= \frac{2n}{e^2}\left(1+\frac{(\ln 4\pi n)}{2n}+O\left(\frac{\ln^2 n}{n^2}\right)\right)\\ &= \frac{2n}{e^2}+\frac{\ln(4\pi n)}{e^2}+O\left(\frac{\ln^2 n}{n}\right).\end{aligned}$$

We conclude that

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{(n!)!(2n-1)!!}}{n}\right)\\ =\ &\lim_{n\to\infty}\left(\frac{2(n+1)}{e^2}+\frac{\ln(4\pi(n+1))}{e^2}-\frac{2n}{e^2}-\frac{\ln(4\pi n)}{e^2}+O\left(\frac{\ln^2 n}{n}\right)\right)\\ =\ &\frac{2}{e^2}+\lim_{n\to\infty}\left(\frac{\ln\left(\frac{n+1}{n}\right)}{e^2}+O\left(\frac{\ln^2 n}{n}\right)\right)=\frac{2}{e^2}.\end{aligned}$$

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

For each positive integer n, we let

$$a_n = \frac{\sqrt[n]{n!(2n-1)!!}}{n} = \frac{1}{n}\sqrt[n]{\frac{n!(2n)!}{2^n \cdot n!}} = \frac{1}{2n}\sqrt[n]{(2n)!}.$$

Next, we apply a version of Stirling's formula due to Robbins [1], namely $n! = \sqrt{2\pi n}(n/e)^n e^{r_n}$, where $1/(12n+1) < r_n < 1/(12n)$. This yields

$$a_n = \frac{(4\pi n)^{1/(2n)}(2n/e)^2 e^{r_{2n}/n}}{2n} = \frac{2n}{e^2}\left(e^{r_{2n}}\sqrt{4\pi n}\right)^{1/n}.$$

Hence

$$\begin{aligned} a_{n+1} - a_n &= \frac{2n+2}{e^2}\left(e^{r_{2n+2}}\sqrt{4\pi n + 4\pi}\right)^{1/(n+1)} - \frac{2n}{e^2}\left(e^{r_{2n}}\sqrt{4\pi n}\right)^{1/n} \\ &= \frac{2n}{e^2}\left[\left(e^{r_{2n+2}}\sqrt{4\pi n+4\pi}\right)^{1/(n+1)} - \left(e^{r_{2n}}\sqrt{4\pi n}\right)^{1/n}\right] + \frac{2}{e^2}\left(e^{r_{2n+2}}\sqrt{4\pi n + 4\pi}\right)^{1/(n+1)}, \end{aligned}$$

so $\lim_{n\to\infty}(a_{n+1} - a_n) = 0 + \frac{2}{e^2} = \frac{2}{e^2}$.

[1] H. Robbins, A remark on Stirling's formula, *The American Mathematical Monthly* 62(1), Jan. 1955, 26-29.

Solution 3 by Adnan Ali (student), A.E.C.S-4, Mumbai, India

Lemma. [1] *If the positive sequence* (p_n) *is such that*

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = p > 0,$$

then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \frac{p}{e}.$$

Taking $p_n = \frac{n!(2n-1)!!}{n^n}$, we have

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = \lim_{n\to\infty}\frac{n^{n-1}(2n+1)}{(n+1)^n} = \lim_{n\to\infty}\frac{2n+1}{n+1}\left(1-\frac{1}{n+1}\right)^{n-1} = 2e^{-1},$$

and so from our Lemma, the required limit evaluates to $2/e^2$.

REFERENCES

[1] Gh. Toader, Lalescu sequences, Publikacije-Elektrotehnickog Fakulteta Univerzitet U Beogradu Serija Matematika, 9 (1998), 1928.

Editor's comment : The authors of this problem, **D. M. Bătinetu-Giurgiu,** and **Neculai Stanciu** proved in their solution the following generalization:

If $t \in R_+^*$ and $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ are positive real sequences such that $\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a \in R_+^*$ and $\lim_{n\to\infty}\frac{b_{n+1}}{n^t b_n} = b \in R_+^*$ then

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_nb_n}}{n^t}\right)=\frac{ab}{e^{t+1}}.$$

Letting $t=1$, $a_n=n!$ and $b_n=(2n-1)!!$, then

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=\lim_{n\to\infty}\frac{(n+1)!}{n\cdot n!}=\lim_{n\to\infty}\frac{n+1}{n}=1 \text{ and } \lim_{n\to\infty}\frac{(2n+1)!!}{n\cdot(2n-1)!!}=\lim_{n\to\infty}\frac{2n+1}{n}=2,$$

I.e., $a=1$ and $b=2$. So

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{n!(2n-1)!!}}{n}\right)=\frac{ab}{e^{t+1}}=\frac{1\cdot 2}{e^{1+1}}=\frac{2}{e^2}.$$

Also solved by Arkady Alt, San Jose, CA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawaskaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

- **5399:** *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let a, b, c be positive real numbers. Prove that

$$\sum_{cyclic}\frac{2a+2b}{\sqrt{6a^2+4ab+6b^2}}\le 3.$$

Solution by Ed Gray, Highland Beach, FL

By symmetry it is sufficient to show that when x and y are positive, real numbers then

$$f(x,y)=\frac{2x+2y}{\sqrt{6x^2+4xy+6y^2}}\le 1.$$

Squaring both sides, is

$$\begin{aligned}(2x+2y)^2 &\le 6x^2+4xy+6y^2? \text{ Or equivalently, is}\\ 0 &\le 2x^2-4xy+2y^2=(2)(x-y)^2? \text{ But this is obviously true.}\end{aligned}$$

Therefore the statement of the problem is true.

Editor's comment : **D.M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania** with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania** generalized the problem as follows:

If $a, b, c, m, n, p \in R_+^*$, then $\displaystyle\sum_{cyclic}\frac{m(a+b)}{\sqrt{(n+2p)a^2+2nab+(n+2p)b^2}}\le\frac{3m}{\sqrt{n+p}}$.

After proving the generalization, they let $m=n=p=2$, obtaining the statement of the problem.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Nikos Kalapodis, Patras, Greece; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawaskaki, Japan; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota "Traian Vuia" Technical College, Focansi,Romania, and the proposer

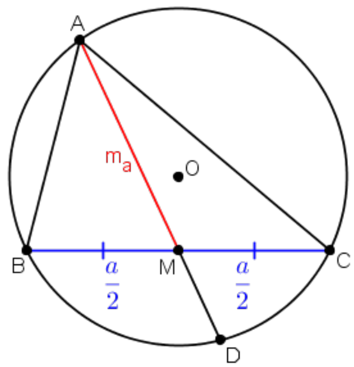
5400: *Proposed by Arkady Alt, San Jose, CA*

Prove the inequality

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\leq 12\,(2R-3r),$$

where a,b,c and m_a,m_b,m_c are respectively sides and medians of $\triangle ABC$, with circumradius R and inradius r.

Solution 1 by Nikos Kalapodis, Patras, Greece



Let the median $AM=m_a$ intersects the circumcircle of triangle ABC at D.

Then by the intersecting chords theorem we have
$AM\cdot MD=MB\cdot MC$ or $AM\cdot(AD-AM)=MB\cdot MC$.

It follows that $m_a\cdot AD-m_a^2=\dfrac{a^2}{4}$ i.e. $\dfrac{a^2}{m_a}=4AD-4m_a$.

By the obvious inequality $AD\leq 2R$ we obtain that $\dfrac{a^2}{m_a}\leq 8R-4m_a$ (1).

Taking into account the other two similar inequalities we have

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\leq 24R-4(m_a+m_b+m_c)\quad (2).$$

Inequality (1) can be rewritten as $m_a\geq\dfrac{a^2+4m_a^2}{8R}$. Adding the other two similar inequalities and using the following well-known identities

$m_a^2+m_b^2+m_c^2=\dfrac{3}{4}(a^2+b^2+c^2)$, $bc=2Rh_a$, $\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}=\dfrac{1}{r}$ we get that

$$m_a+m_b+m_c \geq \frac{a^2+b^2+c^2+4(m_a^2+m_b^2+m_c^2)}{8R} = \frac{a^2+b^2+c^2}{2R} \geq \frac{bc+ca+ab}{2R}$$

$$= h_a+h_b+h_c \geq \frac{9}{\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}}$$

$$= \frac{9}{\dfrac{1}{r}} = 9r, \quad \text{i.e.} \quad m_a+m_b+m_c \geq 9r \quad (3).$$

Combining (2) and (3) we have

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq 24R-4(m_a+m_b+m_c) \leq 24R-4\cdot 9r = 12(2R-3r).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let A', B', C' and $A", B", C"$ be respectively the midpoints of the sides BC, CA, AB and the intersections of the medians AA', BB', CC' with the circumcircle of $\triangle ABC$ and let us denote h_a, h_b, h_c the heights and $n_a = A'A'', n_b = B'B'', n_c = C'C''$

Taking into account that the absolute value of the power of A' with respect to the circumcircle of $\triangle ABC$ is $A'B \cdot A'C$ and also $A'A \cdot A'A''$, that is $\frac{a}{2}\cdot\frac{a}{2} = m_a \cdot n_a$ or equivalently $\frac{a^2}{m_a} = 4n_a$.

Since $m_a + n_a \leq 2R$ (AA' is a chord of the circumcircle whose diameter is $2R$) and $h_a \leq m_a$ (the height is the minimum distance from the vertex to its opposite side), we conclude that $na_a \leq 2R - m_a \leq 2R - h_a$.

Thus $\frac{a^2}{m_a} \leq 4(2R-h_a)$ and analogously $\frac{b^2}{m_b} \leq 4(2R-h_b)$ and $\frac{c^2}{m_c} \leq 4(2R-h_c)$ so

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq 12\left(2R-\frac{1}{3}(h_a+h_b+h_c)\right).$$

The result follows from $h_a+h_b+h_c \geq 9r$, with equality iff $\triangle ABC$ is equilateral which is equality 6.8 from page 61 in the book *Geometric inequalities* by O. Bottema, R. Ž. Djordjević, R.R. Janić, D.S. Mitrinović and P.M Vasić, Wolters Noordhoff, Groningen, 1969.

Equality is attained iff $m_a + n_a = 2R,\ h_a = m_a$ and $\mathrm{h}_a + h_b + h_c = 9r$ and cyclically, that is, iff $\triangle ABC$ is an equilateral triangle.

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Using the inequality $m_a \geq \frac{b^2+c^2}{4R}$, we obtain

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq \sum\frac{4Ra^2}{b^2+c^2}$$

$$\frac{2(2R-3r)}{R} - \sum\frac{a^2}{b^2+c^2} \geq 0 \iff \frac{3(2R-3r)}{R} - \frac{3}{2} \geq 0 \Longrightarrow R \geq 2r, \text{ which is true.}$$

(3)

* Where, using Nesbitt's inequality, we have $\sum \dfrac{a^2}{b^2+c^2} \geq \dfrac{3}{2}$.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawaskaki, Japan, and the proposer.

5401: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c be three positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2}+\frac{c^{-1}}{(4\sqrt{b}+3\sqrt{c})^2}+\frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2} \geq \frac{3}{49}.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The proposed inequality may be written as

$$\frac{1}{\left(4\sqrt{ab}+3b\right)^2}+\frac{1}{\left(4\sqrt{bc}+3c\right)^2}+\frac{1}{(4\sqrt{ca}+3a)^2} \geq \frac{3}{49}.$$

Now, by the Cauchy-Shwartz inequality in Engel form, the left-hand side is

$$\begin{aligned} LHS &\geq \frac{3^2}{\left(4\sqrt{ab}+3b\right)^2+\left(4\sqrt{bc}+3c\right)^2+(4\sqrt{ca}+3a)^2} \\ &= \frac{3^2}{16\,(ab+bc+ca)+9\,(a^2+b^2+c^2)+24\left(b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca}\right)}. \end{aligned}$$

By the rearrangement inequality, $ab+bc+ca \leq a^2+b^2+c^2$ and $b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca} \leq a^2+b^2+c^2$, so

$$LHS \geq \frac{3^2}{(16+9+24)\,(a^2+b^2+c^2)} = \frac{3^2}{(16+9+24)\,3} = \frac{3}{49}$$

with equality if and only if $a=b=c=1$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $x, y > 0$, then two forms of the Arithmetic - Geometric Mean Inequality state that

$$2\sqrt{xy} \leq x+y \quad \text{and} \quad 2xy \leq x^2+y^2.$$

In both cases, equality is attained if and only if $x = y$. As a result, we have

$$\begin{aligned} y\left(4\sqrt{x}+3\sqrt{y}\right)^2 &= y\left(16x+24\sqrt{xy}+9y\right) \\ &\le y\left[16x+12\left(x+y\right)+9y\right] \\ &= 7y\left(4x+3y\right) \\ &= 7\left(4xy+3y^2\right) \\ &\le 7\left[2\left(x^2+y^2\right)+3y^2\right] \\ &= 7\left(2x^2+5y^2\right), \end{aligned} \tag{1}$$

with equality if and only if $x = y$.
We will also need the known result that if $X, Y, Z > 0$, then

$$(X+Y+Z)\left(\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}\right) \ge 9. \tag{2}$$

(This is a direct result of applying the Cauchy - Schwarz Inequality to the vectors $\overrightarrow{V} = \left(\sqrt{X}, \sqrt{Y}, \sqrt{Z}\right)$ and $\overrightarrow{W} = \left(\frac{1}{\sqrt{X}}, \frac{1}{\sqrt{Y}}, \frac{1}{\sqrt{Z}}\right)$.)
By (1), (2), and the constraint equation $a^2 + b^2 + c^2 = 3$,

$$\begin{aligned} &\frac{b^{-1}}{\left(4\sqrt{a}+3\sqrt{b}\right)^2}+\frac{c^{-1}}{\left(4\sqrt{b}+3\sqrt{c}\right)^2}+\frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2} \\ &\ge \frac{1}{7}\left[\frac{1}{2a^2+5b^2}+\frac{1}{2b^2+5c^2}+\frac{1}{2c^2+5a^2}\right] \\ &= \frac{1}{147}\cdot 21\cdot\left[\frac{1}{2a^2+5b^2}+\frac{1}{2b^2+5c^2}+\frac{1}{2c^2+5a^2}\right] \\ &= \frac{1}{147}\left[\left(2a^2+5b^2\right)+\left(2b^2+5c^2\right)+\left(2c^2+5a^2\right)\right]\left[\frac{1}{2a^2+5b^2}+\frac{1}{2b^2+5c^2}+\frac{1}{2c^2+5a^2}\right] \\ &\ge \frac{9}{147} \\ &= \frac{3}{49}, \end{aligned}$$

with equality if and only if $a = b = c = 1$.

Solution 3 by Henry Ricardo, New York Math Circle, NY

The arithmetic-geometric mean (AM-GM) inequality gives us

$$(4\sqrt{a}+3\sqrt{b})^2 = 16a+24\sqrt{ab}+9b \le 16a+24\left(\frac{a+b}{2}\right)+9b = 28a+21b.$$

Then, using the Cauchy-Schwarz inequality and the AM-GM inequality, we see that

$$\begin{aligned} \sum_{cyclic}\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2} &\ge \sum_{cyclic}\frac{b^{-1}}{28a+21b} = \sum_{cyclic}\frac{1}{28ab+21b^2} \\ &\ge \frac{(1+1+1)^2}{\sum_{cyclic}(28ab+21b^2)} = \frac{9}{28(ab+bc+ca)+21(a^2+b^2+c^2)} \\ &\ge \frac{9}{28(a^2+b^2+c^2)+21(a^2+b^2+c^2)} = \frac{9}{49(3)} = \frac{3}{49}. \end{aligned}$$

Equality holds if and only if $a=b=c=1$.

Solution 4 by Toshihiro Shimizu, Kawaskaki, Japan

From Cauchy-Schwartz's inequality,

$$\left(a^2+b^2+c^2\right)\left(\frac{b^{-1}}{(4\sqrt{a}+3\sqrt{b})^2}+\frac{c^{-1}}{(4\sqrt{b}+3\sqrt{c})^2}+\frac{a^{-1}}{(4\sqrt{c}+3\sqrt{a})^2}\right)\geq\left(\sum_{cyclic}\frac{a}{\sqrt{b}(4\sqrt{a}+3\sqrt{b})}\right)^2$$
$$=\left(\sum_{cyclic}\frac{1}{4\sqrt{\frac{b}{a}}+3\cdot\frac{b}{a}}\right)^2$$

Let $x=\log\left(\sqrt{\dfrac{b}{a}}\right)$, $y=\log\left(\sqrt{\dfrac{c}{b}}\right)$, $z=\log\left(\sqrt{\dfrac{a}{c}}\right)$.Then, $x+y+z=0$. The (r.h.s) of the above inequality is equal to

$$\left(\sum_{cyclic}\frac{1}{4e^x+3e^{2x}}\right)^2$$

Let $f(x)=1/\left(4e^x+3e^{2x}\right)$. Since $f''(x)=4e^{-x}\left(9e^x+9e^{2x}+4\right)/\left(3e^x+4\right)^3>0$, f is convex. Thus, from Jensen's inequality, it follows that

$$\begin{aligned}f(x)+f(y)+f(z)&\geq 3f\left(\frac{x+y+z}{3}\right)\\&=3f(0)\\&=\frac{3}{7}\end{aligned}$$

Solution 5 by David E. Manes, SUNY Oneonta, Oneonta, NY

Let

$$L=\sum_{cyclic}\frac{b^{-1}}{(4\sqrt{2}+3\sqrt{b})^2}=\sum_{cyclic}\frac{1}{b(4\sqrt{a}+3\sqrt{b})^2}.$$

Define vectors $\vec{u}$ and $\vec{v}$ such that

$$\begin{aligned}\vec{u}&=\left(\frac{1}{\sqrt{b}(4\sqrt{a}+3\sqrt{b})},\frac{1}{\sqrt{c}(4\sqrt{b}+3\sqrt{c})},\frac{1}{\sqrt{a}(4\sqrt{c}+3\sqrt{a})}\right).\\ \vec{v}&=\left(\sqrt{b}(4\sqrt{a}+3\sqrt{b}),\sqrt{c}(4\sqrt{b}+3\sqrt{c}),\sqrt{a}(4\sqrt{c}+3\sqrt{a})\right).\end{aligned}$$

Then the dot product of $\vec{u}$ and $\vec{v}$ is less than or equal to the product of the norms of $\vec{u}$ and $\vec{v}$ by the Cauchy-Schwarz inequality. Therefore,

$$1+1+1\leq\sqrt{\sum_{cyclic}\frac{1}{b(4\sqrt{a}+3\sqrt{b})^2}}\sqrt{\sum_{cyclic}b(4\sqrt{a}+3\sqrt{b})^2}$$

or

$$L \geq \frac{9}{\sum_{cyclic} b(4\sqrt{a}+3\sqrt{b})^2}.$$

Expanding the denominator, one obtains

$$\sum_{cyclic} b(4\sqrt{a}+3\sqrt{v})^2 = 16\left(\sum_{cyclic} ab\right) + 24\left(\sum_{cyclic} \sqrt{ab^3}\right) + 9(a^2+b^2+c^2).$$

The Rearrangement inequality implies

$$\sum_{cyclic} ab + \sum_{cyclic} \sqrt{ab^3} \leq (a^2+b^2+c^2) + \left(\sqrt{a^4}+\sqrt{b^4}+\sqrt{c^4}\right)$$

with equality if and only if $a=b=c$. Therefore,

$$\frac{1}{\sum_{cyclic} b(4\sqrt{a}+3\sqrt{b})^2} \geq \frac{1}{16\sum_{cyclic} a^2 + 24\sum_{cyclic} a^2 + 9\sum_{cyclic} a^2}.$$

Since $a^2+b^2+c^2=3$, it follows that

$$\frac{1}{\sum_{cyclic} b(4\sqrt{a}+3\sqrt{b})^2} \geq \frac{1}{16(3)+24(3)+9(3)} = \frac{1}{3(49)}.$$

Hence,

$$L = \sum_{cyclic} \frac{b^{-1}}{(4\sqrt{2}+3\sqrt{b})^2} \geq \frac{9}{3(49)} = \frac{3}{49}$$

with equality if and only if $a=b=c=1$.

Editor′s comment : **D.M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania** with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania** generalized the problem as follows:

If $a,b,c,m,n \in R_+^*$, then $\sum_{cyclic} \frac{b^{-1}}{(m\sqrt{a}+n\sqrt{b})^2} \geq \frac{3}{(n+p)^2}$.

They did this by showing that

$$\sum_{cyclic} \frac{b^{-1}}{(m\sqrt{a}+n\sqrt{b})^2} \geq \frac{27}{(m+n)^2(a+b+c)^2}. \quad (2)$$

Then they used the hypothesis concluding that

$$3 = a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3} \iff 9 \geq (a+b+c)^2. \quad (3)$$

By (2) and (3) they obtained

$$\sum_{cyclic} \frac{b^{-1}}{(m\sqrt{a}+n\sqrt{b})^2} \geq \frac{27}{(m+n)^2(a+b+c)^2} = \frac{3}{(m+n)^2}.$$

Letting $m = 1$and $n = 3$ they obtained the statement of the problem.

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Nikos Kalapodis, Patras, Greece; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department of Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5402: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^\infty \left(\frac{\cos(ax) - \cos(bx)}{x}\right)^2 \mathrm{d}x,$$

where a and b are real numbers.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Obviously we may assume $a \neq b$, since otherwise the integral is null. Let us suppose that $a > b > 0$. Using parity, write the integral as

$$I = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{\cos(ax) - \cos(bx)}{x}\right)^2 dx,$$

and then deform the contour to be the line C slightly below the real axis. Next express cosines in terms of exponentials. Then we obtain I equal to

$$\frac{1}{8}\left(\int_C \frac{-2\left(e^{-(a+b)xi} + e^{(a-b)xi} + e^{(b-a)xi} + e^{(a+b)xi}\right) + e^{-2axi} + e^{2axi} + e^{-2bxi} + e^{2bxi} + 4}{x^2} dx\right).$$

For $a > b > 0$, in the integrals containing terms of the form e^{-kxi}, with $k > 0$, the contour can be closed in the lower half plane (by Jordan lemma) and therefore these integrals vanish (as there are no singularities inside).

The integrals containing terms of the form e^{kxi}, with $k \geq 0$, can only be closed in the upper half plane and are therefore given by the residues at $x = 0$. Therefore

$$\begin{aligned} I &= \frac{\pi i}{4}\, Res_{x=0}\left(\frac{-2e^{(a-b)xi} - 2e^{(a+b)xi} + e^{2axi} + e^{2bxi} + 4}{x^2}\right) \\ &= \frac{\pi i}{4}\left(-2i(a-b) - 2i(a+b) + 2ia + 2ib\right) \\ &= \frac{\pi i 2(-a+b)i}{4} = \frac{\pi(a-b)}{2}. \end{aligned}$$

Solution 2 by Toshihiro Shimizu, Kawaskaki, Japan

For real number $a \neq 0$, we have

$$\begin{aligned}
\int_0^\infty \frac{\sin^2 ax}{x^2} dx &= \frac{1}{2} \int_{-\infty}^\infty \frac{\sin^2 ax}{x^2} dx \\
&= \frac{a^2}{2} \int_{-\infty}^\infty \frac{\sin^2 y}{y^2} \frac{dy}{|a|} \text{ (where } y = |a|\, x) \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \frac{\sin^2 y}{y^2} dy \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \sin^2 y \left(-\frac{1}{y}\right)' dx \\
&= \frac{|a|}{2} \left[\sin^2 y \left(-\frac{1}{y}\right)\right]_{-\infty}^\infty - \frac{a}{2} \int_{-\infty}^\infty 2 \sin y \cos y \left(-\frac{1}{y}\right) dy \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \frac{\sin 2y}{y} dy \\
&= \frac{|a|}{2} \int_{-\infty}^\infty \frac{\sin y}{y} dy \\
&= \frac{|a|\, \pi}{2}.
\end{aligned}$$

For $a = 0$, the value of l.h.s is 0 and r.h.s is also 0. Thus, this result is true for any real number a. Then, since

$$\begin{aligned}
(\cos ax - \cos bx)^2 &= \cos^2 ax + \cos^2 bx - 2 \cos ax \cos bx \\
&= 1 - \sin^2 ax + 1 - \sin^2 bx - \cos(ax + bx) - \cos(ax - bx) \\
&= 2 - \sin^2 ax - \sin^2 bx \\
&\quad - \left(1 - 2\sin^2\left(\frac{ax + bx}{2}\right)\right) - \left(1 - 2\sin^2\left(\frac{ax - bx}{2}\right)\right) \\
&= -\sin^2 ax - \sin^2 bx + 2\sin^2\left(\frac{ax + bx}{2}\right) + 2\sin^2\left(\frac{ax - bx}{2}\right),
\end{aligned}$$

it follows that

$$\begin{aligned}
\int_0^\infty \left(\frac{\cos ax - \cos bx}{x}\right)^2 dx &= -\int_0^\infty \frac{\sin^2 ax}{x^2} dx - \int_0^\infty \frac{\sin^2 bx}{x^2} dx \\
&\quad + 2\int_0^\infty \frac{\sin^2\left(\frac{ax+bx}{2}\right)}{x^2} dx + 2\int_0^\infty \frac{\sin^2\left(\frac{ax-bx}{2}\right)}{x^2} dx \\
&= -\frac{|a|\, \pi}{2} - \frac{|b|\, \pi}{2} + 2 \cdot \frac{|a + b|}{4}\pi + 2 \cdot \frac{|a - b|}{4}\pi \\
&= \frac{1}{2}\left(-|a| - |b| + |a + b| + |a - b|\right).
\end{aligned}$$

If a, b are the same sign or 0, we have $-|a| - |b| + |a + b| = 0$ and the answer is $\frac{|a-b|}{2}$, if a, b are the opposite sign, $-|a| - |b| + |a - b| = 0$ and the answer is $\frac{|a+b|}{2}$.
Also, we can write this answer as

$$\min\left\{\frac{|a - b|}{2}, \frac{|a + b|}{2}\right\}.$$

Solution 3 by Ed Gray, Highland Beach, FL

In order to calculate: $\displaystyle\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 \mathrm{d}x$, where a and b are real numbers we first expand the numerator so that the integral becomes

$$\int_0^\infty \frac{\cos^2(ax) - 2\cos(ax)\cos(bx) + \cos^2(bx)}{x^2}\mathrm{d}x. \qquad (1)$$

But the expression $2\cos(ax)\cos(bx) = \cos(ax+bx) + \cos(ax-bx)$, so equation (1) becomes

$$\int_0^\infty \frac{\cos^2(ax)}{x^2} - \int_0^\infty \frac{\cos(ax+bx)}{x^2} - \int_0^\infty \frac{\cos(ax-bx)}{x^2} + \int_0^\infty \frac{\cos^2(bx)}{x^2}$$

We evaluate each of these four integrals.

We may use "integration by parts" and other standard procedures to obtain the following:

$$\int_0^\infty \frac{\cos^2(ax)}{x^2} = \frac{-a\pi}{2}$$

$$\int_0^\infty \frac{-\cos(ax+bx)}{x^2} = \frac{(a+b)\pi}{2}$$

$$\int_0^\infty \frac{-\cos(ax-bx)}{x^2} = \frac{(a-b)\pi}{2} \text{ if } a > b; \ = \frac{(b-a)\pi}{2} \text{ if } b > a.$$

$$\int_0^\infty \frac{\cos^2(bx)}{x^2} = \frac{-b\pi}{2}$$

Summing the four integrals above we see that

$$\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 \mathrm{d}x = \begin{cases} \dfrac{(a-b)\pi}{2}, & \text{if } b < a \\ \dfrac{(b-a)\pi}{2}, & \text{if } a < b. \end{cases}$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland

We claim that $f(a,b) = \displaystyle\int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 dx = \frac{\pi}{2}\,||b|-|a||$.

Obviously, $f(a,b) = f(b,a) = -f(-a,b) = f(a,-b)$. (1)

Let $r>0$ and let L be the "indented" line: $-\infty < t \le -r,\ re^{i\varphi},\ \pi \ge \varphi \ge 0,\ r \le t < \infty$, run through "from left to right". Let a be a real number. Then $\displaystyle\int_L \frac{e^{iaz}}{z^2}dz = \pi\,(a\,|a|)$.

Indeed, By Cahuchy's theorem, the integral does not end on r. Assume that $a \ge 0$. Then

$$\left|\int_L \frac{e^{iaz}}{z^2}dz\right| \le 2\int_r^\infty \frac{1}{t^2}dt + \frac{\pi r}{r^2}\max_{0\le\varphi\le\pi}\left|e^{iare^{i\varphi}}\right| = \frac{1}{r}(2+\pi) \to 0,\ as\ r \to \infty.$$

So $\int_L \frac{e^{iaz}}{z^2}dz = 0$, if $a \geq 0$, where $\overline{L}$ is the complex conjugate of L, i.e., the line L reflected at the abscissa

By the residue theorem,

$$\int_{\overline{L}} \frac{e^{iaz}}{z^2}dz - \int_L \frac{e^{iaz}}{z^2}dz = \int_{|z|=r} \frac{e^{iaz}}{z^2}dz = 2\pi i Res\left(\frac{e^{iaz}}{z^2},\ z=0\right) = -2\pi a.$$

So $\int_L \frac{e^{iaz}}{z^2}dz = \int_{\overline{L}} \frac{e^{iaz}}{z^2}dz - 2\pi i Res\left(\frac{e^{iaz}}{z^2},\ z=0\right) = 2\pi a,$ if $a < 0$.

To sum up:

$$\int_L \frac{e^{iaz}}{z^2}dz = \begin{cases} 0, & a \geq 0 \\ 2\pi a, & a < 0. \end{cases} = \pi\left(a - |a|\right), \text{as claimed.}$$

We conclude that

$$f(ab) = \int_0^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 dx = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{\cos(ax)-\cos(bx)}{x}\right)^2 dx = \frac{1}{2}\int_L \left(\frac{\cos(az)-\cos(bz)}{z}\right)^2 dz =$$

$$= \frac{1}{2}\int_L \frac{\cos^2(az) + -2\cos(az)\cos(bz) + \cos^2(bz)}{z^2}dz$$

$$= \frac{1}{2}\int_L \frac{\left(e^{iaz}+e^{-iaz}\right)^2 - 2\left(e^{iaz}+e^{-iaz}\right)\left(e^{iaz}+e^{-iaz}\right) + \left(e^{iaz}+e^{-iaz}\right)^2}{4z^2}dz$$

$$= \frac{1}{2}\int_L \frac{e^{2iaz}+e^{2ibz}+e^{-2iaz}+e^{-2ibz}+4-2e^{i(a+b)z}-2e^{i(a-b)z}-2e^{i(-a+b)z}-2e^{i(-a-b)z}}{4z^2}dz$$

$$\begin{aligned} &= \frac{\pi}{8}\Big(2a-|2a|+2b-|2b|-2a-|2a|-2b-|2b|-2(a+b) \\ &\quad +2|a+b|-2(a-b)+2|a-b|-2(-a+b)+2|-a+b|-2(-a-b)+2|-a-b|\Big) \\ &= \frac{\pi}{4}\Big(-|a|-|b|-|a|-|b|+|a+b|+|a-b|+|-a+b|+|-a-b|\Big) \\ &= \frac{\pi}{2}\Big(-|a|-|b|+|a+b|+|a-b|\Big). \end{aligned}$$

By (1) we can assume that $0 \leq a \leq b$. Then

$$\begin{aligned} f(a,b) &= \frac{\pi}{2}\left(-|a|-|b|+|a+b|+|a-b|\right) \\ &= \frac{\pi}{2}\left(-a-b+a+b+b-a\right) \end{aligned}$$

$$\begin{aligned} &= \frac{\pi}{2}(b-a) \\ &= \frac{\pi}{2}\big||b|-|a|\big|, \text{ as claimed.} \end{aligned}$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

Denote the given integral by I. We show that

$$I = \frac{(|a+b|+|a-b|-|a|-|b|)\,\pi}{2} \qquad (1)$$

4pt It is well known that for any real number r, we have

$$\int_0^\infty \frac{\sin(rx)}{x}dx = \begin{cases} \pi/2 & r>0 \\ 0 & r=0. \\ -\pi/2 & r<0. \end{cases} \qquad (2)$$

Since $\lim\limits_{x\to 0}\dfrac{(\cos(ax)-\cos(bx))^2}{x} = 0$, so intergrating by parts , we obtain

$$\begin{aligned} I &= \int_0^\infty \frac{f(a,b,x)}{x}dx, \text{ where} \\ & f(a,b,x) \\ &= 2\left(\cos(ax)-\cos(bx)\right)\left(b\sin(bx)-a\sin(ax)\right) \\ &= 2b\sin(bx)\cos(ax)+2a\sin(ax)\cos(bx)-a\sin(2ax)-b\sin(2bx) \\ &= (a+b)\sin\left((a+b)x\right)+(a-b)\sin\left((a-b)x\right)-a\sin(2ax)-b\sin(2bx). \end{aligned}$$

Using (2), we now obtain (1). This completes the proof.

Solution 6 by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

We prove that the value of the proposed integral is $(a-b)\dfrac{\pi}{2}$. It is trivial when $a=b$, so we assume that $a \neq b$. We make repeated use of the following integral (proof of which is provided at the end, for the sake of completion)

$$\int_0^\infty e^{-\alpha x}\cos(\beta x)dx = \frac{\alpha}{\alpha^2+\beta^2}$$

We have the identity (easily verified) $\dfrac{1}{x^2} = \displaystyle\int_0^\infty te^{-xt}dt$. Using this, the proposed integral becomes

$$\int_0^\infty \int_0^\infty te^{-xt}(\cos(ax)-\cos(bx))^2 dtdx.$$

Since everything is positive, by Tonelli's Theorem, we can reverse the order of integration so that the integral now becomes

$$\int_0^\infty \int_0^\infty te^{-xt}(\cos(ax) - \cos(bx))^2 dxdt.$$

From the trigonometric identities $\dfrac{\cos(2x)+1}{2} = \cos^2(x)$ and $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$, we easily obtain (using (1))

$$\int_0^\infty e^{-xt}(\cos^2(ax) + \cos^2(bx))dx = \frac{1}{t} + \frac{1}{2}\left(\frac{t}{t^2+(2a)^2} + \frac{t}{t^2+(2b)^2}\right)$$

and

$$\int_0^\infty e^{-xt}(2\cos(ax)\cos(bx))dx = \frac{t}{t^2+(a+b)^2} + \frac{t}{t^2+(a-b)^2}.$$

Thus, (2) becomes

$$\int_0^\infty \int_0^\infty te^{-xt}(\cos(ax) - \cos(bx))^2 dxdt$$

$$= \int_0^\infty t\left(\frac{1}{t} + \frac{1}{2}\left(\frac{t}{t^2+(2a)^2} + \frac{t}{t^2+(2b)^2}\right) - \frac{t}{t^2+(a+b)^2} - \frac{t}{t^2+(a-b)^2}\right)dt$$

$$= \left[(a-b)\arctan\frac{t}{a-b} + (a+b)\arctan\frac{t}{a+b} - a\arctan\frac{t}{2a} - b\arctan\frac{t}{2b}\right]_{t=0}^{t=\infty} = (a-b)\frac{\pi}{2}.$$

Proof of (1):

Let $I = \int_0^\infty e^{-\alpha x}\cos(\beta x)dx$. Then $\cos(\beta x) = \dfrac{e^{i\beta x}+e^{-i\beta x}}{2}$ gives

$$I = \frac{1}{2}\int_0^\infty \left(e^{-x(\alpha - i\beta)} + e^{-x(\alpha+i\beta)}\right)dx = \frac{1}{2}\left(\frac{1}{\alpha - i\beta} + \frac{1}{\alpha+i\beta}\right) = \frac{\alpha}{\alpha^2+\beta^2}.$$

Alternatively, one can do integration by parts to get the same result.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

Mea Culpa

Paolo Perfetti of the Mathematics Department of Tor Vergata University in Rome, Italy should have been credited with having solved 5394.