Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
February 15, 2017

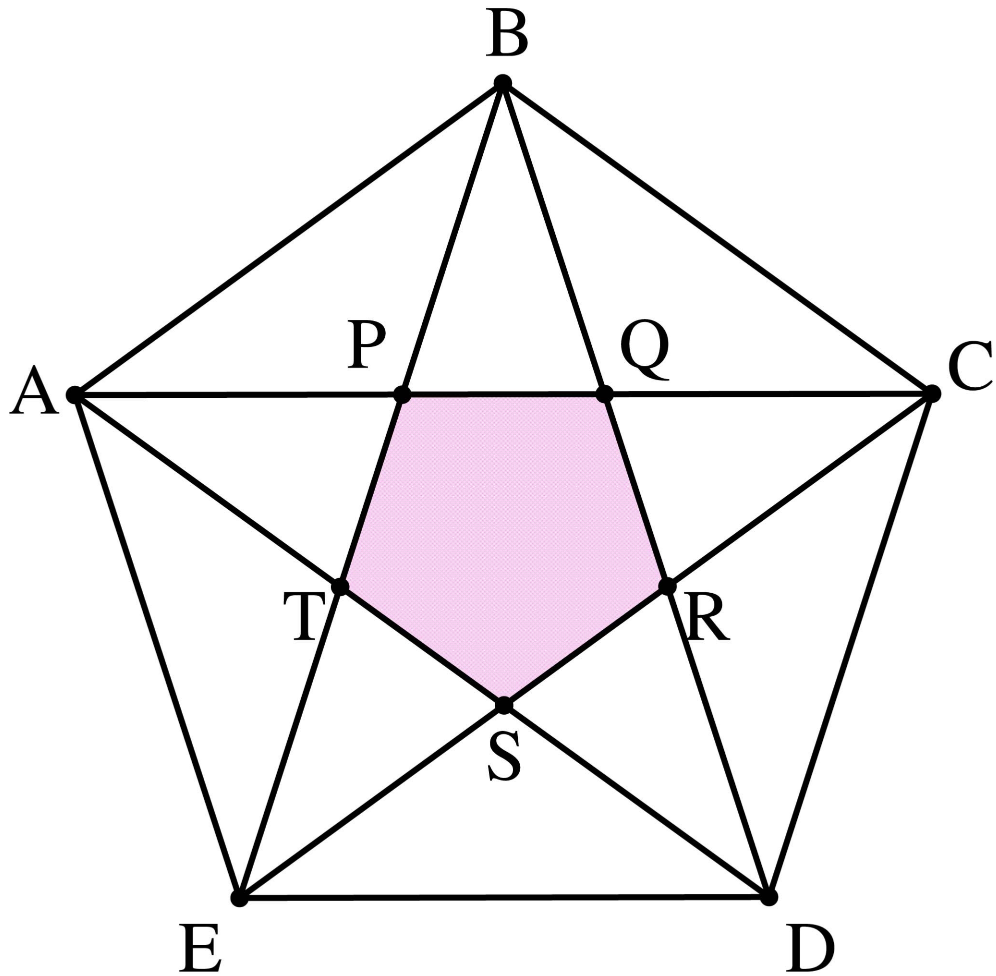
- **5421:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $1-d$ bisects two of its sides.

- **5422:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Polygon $ABCDE$ is a regular pentagon. Pentagon $PQRST$ is bounded by diagonals of pentagon $ABCDE$ as shown. Find the following:

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE}.$$



- **5423:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

  Let a, b, c be the side-lengths, r_a, r_b, r_c be the radii of the ex-circles and R, r the radii of the circumcircle and incircle respectively, and s the semiperimeter of $\triangle ABC$. Show that

$$\frac{(r_a - r)^2 + r_b r_c}{(s-b)(s-c)} + \frac{(r_b - r)^2 + r_c r_a}{(s-c)(s-a)} + \frac{(r_c - r)^2 + r_a r_b}{(s-a)(s-b)} \geq 13.$$

- **5424:** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

  Let a, b, c and d be positive real numbers such that $abc + bcd + cda + dab = 4$. Prove that $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) \geq 256$.

- **5425:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$. If n is an odd positive integer then show that $1 + \det(A)$ is the product of two consecutive Fibonacci numbers, where

$$A = \begin{pmatrix} F_1^2 - 1 & F_1F_2 & F_1F_3 & \dots & F_1F_n \\ F_2F_1 & F_2^2 - 1 & F_2F_3 & \cdots & F_2F_n \\ F_3F_1 & F_3F_2 & F_3^2 - 1 & \cdots & F_3F_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_nF_1 & F_nF_2 & F_nF_3 & \cdots & F_n^2 - 1 \end{pmatrix}$$

5426: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of natural numbers. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]} \text{ converges.}$$

Here $[x, y]$ denotes the least common multiple of the natural numbers x and y.

Solutions

- **5403:** *Proposed by Kenneth Korbin, New York, NY*

Let $\phi = \dfrac{1+\sqrt{5}}{2}$. Solve the equation $\sqrt[3]{x+\phi} = \sqrt[3]{\phi} + \sqrt[3]{x-\phi}$ with $x > \phi$.

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak, Angelo State University, San Angelo, TX

Let $a = \sqrt[3]{x+\phi}$ and $b = \sqrt[3]{x-\phi}$. We may write

$$\begin{aligned} a - b &= \sqrt[3]{\phi} \\ (a-b)^3 &= \phi \\ a^3 - 3a^2b + 3ab^2 - b^3 &= \phi \\ a^3 - b^3 - 3ab(a-b) &= \phi \\ x + \phi - (x - \phi) - 3\sqrt[3]{x^2 - \phi^2}\sqrt[3]{\phi} &= \phi. \end{aligned}$$

Simplifying this last equation we obtain $\sqrt[3]{x^2 - \phi^2} = \dfrac{\phi^{2/3}}{3}$. Under the condition $x > \phi$, the solution to this equation is $x = \sqrt{\dfrac{\phi^2}{27} + \phi^2} = \dfrac{2\sqrt{21}}{9}\phi$.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC.

Given any real number $a > 0$, we solve the equation $\sqrt[3]{x+a} = \sqrt[3]{a} + \sqrt[3]{x-a}$ with $x > a$. (Similarly, given any real number $a < 0$, we may solve the equation $\sqrt[3]{x+a} = \sqrt[3]{a} + \sqrt[3]{x-a}$ with $x < a$.)

Rewriting the given equation and cubing both sides yields

$$(x+a) - 3\sqrt[3]{(x+a)^2(x-a)} + 3\sqrt[3]{(x+a)(x-a)^2} - (x-a) = a,$$

or $3\sqrt[3]{x^2-a^2}(\sqrt[3]{x-a} - \sqrt[3]{x+a}) = -a$. Then $-3\sqrt[3]{a}\sqrt[3]{x^2-a^2} = -a$, so cubing once more produces

$$-27a(x^2-a^2) = -a^3.$$

Hence $x^2 = \dfrac{28}{27}a^2$, so requiring $x > a$ yields $x = \dfrac{2\sqrt{21}}{9}a$. In particular, when $a = \phi$, we obtain the solution $x = \dfrac{2\sqrt{21}}{9}\phi = \dfrac{\sqrt{21}+\sqrt{105}}{9}$.

Solution 3 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The value of $x > \phi$ that satisfies the equation is

$$x = \phi\Big[\Big(\frac{-3+\sqrt{21}}{6}\Big)^3 + 1\Big] \approx 1.48363835038.$$

One notes that $x > \phi$ and does satisfy the equation.
Let $v = \sqrt[3]{x-\phi}$. Then $v^3 = x - \phi$ so that $x = v^3 + \phi$. Since we want the solution $x > \phi$, it follows that x must be positive. The original equation in terms of v is

$$\sqrt[3]{x+2\phi} = \sqrt[3]{\phi} + v.$$

Cubing both sides of this equation, we get

$$3\sqrt[3]{\phi}\cdot v^2 + 3(\sqrt[3]{\phi})^2 v - \phi = 0.$$

Dividing by $3\sqrt[3]{\phi}$ reduces this equation to the monic quadratic equation

$$v^2 + \sqrt[3]{\phi}\cdot v - \frac{1}{3}(\sqrt[3]{\phi})^2 = 0$$

with roots

$$v = \frac{-\sqrt[3]{\phi} \pm \sqrt[3]{\phi}\cdot\sqrt{\frac{7}{3}}}{2}.$$

Rejecting the negative root yields

$$v = \frac{-\sqrt[3]{\phi} + \sqrt[3]{\phi}\cdot\sqrt{\frac{7}{3}}}{2} = \sqrt[3]{\phi}\Big(\frac{-3+\sqrt{21}}{6}\Big).$$

Hence,

$$x = v^3 + \phi = \phi\Big[\Big(\frac{-3+\sqrt{21}}{6}\Big)^3 + 1\Big] = \frac{2\sqrt{21}}{9}\phi.$$

Editor's comment : **D. M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania** with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania** generalized the problem as follows:

Let $a, b, c, > 0$, with $a + b = 2c$ then it can be shown that the unique real-valued solution to the equation $\sqrt[3]{x+a} = \sqrt[3]{x-b} + \sqrt[3]{c}$, where $x > c$ is $x = \dfrac{3\sqrt{3}\,(b-a) + 4c\sqrt{7}}{6\sqrt{3}}$.

If $a = b = \phi$, then $= \phi$ and the equation $\sqrt[3]{x+\phi} = \sqrt[3]{x-\phi} + \sqrt[3]{\phi}$ with $x > \phi$, has the solution

$$x = \frac{3\sqrt{3}\left(\phi - \phi\right) + 4\phi\sqrt{7}}{6\sqrt{3}} = \frac{2\sqrt{21}}{9}\phi.$$

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Ashland University Undergraduate Problem Solving Group, Ashland, OH; D. M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; the proposer, and Students from Taylor University (see below);**

Students at Taylor University, Upland, IN.
Group 1. Ben Byrd, Maddi Guillaume, and Makayla Schultz.
Group 2. Caleb Knuth, Michelle Franch and Savannah Porter.
Group 3. Lauren Moreland, Anna Souzis, and Boni Hermandez

- **5404:** *Proposed Arkady Alt, San Jose, CA*

For any given positive integer $n \geq 3$, find the smallest value of the product of $x_1x_2\ldots x_n$, where $x_1, x_2, x_3, \ldots x_n > 0$ and $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1.$

Solution 1 by Ed Gray, Highland Beach, FL

Suppose each term had the value of $\dfrac{1}{n}$. Since there are n terms, the sum is equal to 1, satisfying the problem restriction.

In the event for each $k, 1 \leq k \leq n$

1. $\dfrac{1}{1+x_k} = \dfrac{1}{n}$, so $x_k = n-1$, and the value of the product is:
2. $(n-1)^n$.

If this is not the smallest product, at least one value of x_k must be less than $n-1$. Suppose $x_k = n-1-e$ where $e > 0$.

Then the series contains the therm $\dfrac{1}{1+x_k} = \dfrac{1}{n-e}$. We must increase the value of another term so that the sum maintains the value of 1. We must have:

3. $\dfrac{1}{n-e} + \dfrac{1}{1+x_m} = \dfrac{2}{n}$
4. $\dfrac{1}{1+x_m} - \dfrac{2}{n} - \dfrac{1}{n-e} = \dfrac{2(n-e-n)}{n(n-e)} = \dfrac{2n-2e-n}{n(n-e)}$
5. $\dfrac{1}{1+x_m} = \dfrac{n-2e}{n(n-e)}$
6. $(1+x_m)(n-2e) = n(n-e)$
7. $1 + x_m = \dfrac{n(n-e)}{n-2e}$

8. $x_m = \dfrac{n(n-e)}{n-2e} - 1 = \dfrac{n(n-e)-n-2e)}{n-2e} = \dfrac{n^2-ne-n+2e}{n-2e}$

9. The new product is: $\Big((n-1)^{n-2}\Big) x_k x_m$. If the new product is to be smaller, we must have:

10. $\dfrac{(n-1)^{n-2}(n-1-e)(n^2-n-e(n-2)}{n-2e} < (n-1)^n$, or dividing by $(n-1)^{n-2}$

11. $(n-1-e)(n^2-n-en+2e) < (n-2e)(n-1)^2$,

12. $(n-1-e)(n^2-n-en+=2e) < (n-2e)(n^2-2n+1)$, which simplifies to:

13. $2en^2+ne2 < 2e^2$. Dividing by e^2,

14. $\dfrac{2n^2}{e}+n < 2$, which is a contraction. Therefore, we did not decrease the product, but increased it.

So $(n-1)^n$ is the minimum product.

Solution 2 by Ramya Dutta (student), Chennai Mathematical Institute) India

Consider the polynomial $P(x) = \prod_{j=1}^{n}(x+x_j)$, then $\dfrac{P'(x)}{P(x)} = \sum_{j=1}^{n}\dfrac{1}{x+x_j}$, i.e., $P'(1) = P(1)$.

Denoting the j-th symmetric polynomial by, $\sigma_j = \sum_{1\le k_1<k_2<\cdots<k_j\le n} x_{k_1}x_{k_2}\cdots x_{k_j}$ for $j \ge 1$ and $\sigma_0 = 1$,

$P(x) = \sum_{j=0}^{n}\sigma_j x^{n-j}$ and $P'(x) = \sum_{j=0}^{n-1}(n-j)\sigma_j x^{n-j-1}$

Therefore, the condition $P(1) = P'(1)$ is equivalent to,

$$\sigma_n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j$$

Using, AM-GM inequality: $\sigma_j \ge \binom{n}{j}\sigma_n^{j/n}$ for $j \ge 1$.

I.e., writing $\sigma_n^{1/n} = \alpha$, we have,

$$\begin{aligned}
\alpha^n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j &\ge \sum_{j=0}^{n-1}(n-j-1)\binom{n}{j}\alpha^j \\
&= (n-1)\sum_{j=0}^{n-1}\binom{n}{j}\alpha^j - n\sum_{j=1}^{n-1}\binom{n-1}{j-1}\alpha^j \\
&= (n-1)\left((1+\alpha)^n-\alpha^n\right) - n\alpha\left((1+\alpha)^{n-1}-\alpha^{n-1}\right) \\
&= \alpha^n - (1+\alpha)^n + n(1+\alpha)^{n-1}
\end{aligned}$$

that is, $(1+\alpha)^n \ge n(1+\alpha)^{n-1} \implies \alpha \ge n-1$ (since, $\alpha > 0$)
So, the minimum value of $x_1x_2\cdots x_n$ is $(n-1)^n$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We shall use the Method of Lagrange Multipliers to show that the smallest value of the product is $(n-1)^n$, achieved when each $x_i = n-1$.

First suppose that all but one of the x_i are equal: let $x_i = b$ for $1 \le i \le n-1$ and choose x_n so that the constraint $\sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1$ is satisfied:

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = (n-1)\frac{1}{1+b} + \frac{1}{1+x_n} = 1, \Longrightarrow x_n = \frac{n-1}{b-(n-2)}, \text{ where}$$

$b > n-2$ to make $x_n > 0$.

Then the product $f(x_1, x_2, \ldots, x_)n) = \prod_{i=1}^{n} x_i = b^{n-1}\frac{n-1}{b-(n-2)}$.

We note that as b becomes unbounded positive, the product of the $x_i's$ becomes unbounded positive, and as b approaches $n-2$ from above, the product of the $x_i's$ also becomes unbounded positive. Thus if the product has an absolute extremum subject to the given constraint, it must be a minimum since the product is unbounded above.

For $b = n-1$, we see that x$_n = n-1$, so every $x_i = n-1$ and the product is equal to $(n-1)^n$,

We consider this as a Lagrange Multiplier problem where we minimize the product $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$ subject to the constraint

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1.$$

That is, subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1.$$

By the Method of Lagrange Multipliers, we'll find the minimum of f where

$\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_n) = \lambda \frac{\partial}{\partial x_i} g(x_1, x_2, \ldots, x_n)$ for $1 \le k \le n$.

We see that: $\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_n) = \prod_{\substack{i=1 \\ i=k}}^{n} x_i$ and $\frac{\partial}{\partial x_i} g(x_1, x_2, \ldots, x_n) = \frac{1}{(1+x_i)^2}$ for $1 \le k \le n$.

Thus we want to solve the system, $\prod_{\substack{i=1 \\ i=k}}^{n} x_i = \frac{\lambda}{(1+x_k)^2}$, for $1 \le k \le n$.

Solving each equation for λ gives $\lambda = -(1+x_k)^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} x_i$ for $1 \le k \le n$.

Hence, for any $1 \le j,\ k \le n$ we must have $\lambda = -(1+x_i)^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} x_i = -(1+x_j)^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} x_i$

Algebra gives $\dfrac{x_j}{(1+x_j)^2} = \dfrac{x_k}{(1+x_k)^2}, \quad 1 \le j,\ k \le n.$

We claim this forces $x_i = x_k$. Suppose that $x_k \ne x_i$ for some $k \ne j$.

Now consider the function $h(x) = \dfrac{x}{(1+x)^2}$ for $x > 0$.

Note that $h(x_i) = h(x_k)$ for $1 \le j,\ k \le n$

By calculus, $h(x)$ is strictly increasing for $0 < x < 1$ to a maximum (of $1/4$) at $x = 1$, and is then strictly decreasing for $x > 1$. That is, h except for the peak at $x = 1$ is two- to- one function (for $x > 0$).

Moreover, $h(x)$ has the reflective property $h\left(\dfrac{1}{x}\right) = h(x)$. Hence, for $1 \le j \ne k \le n,\ h(x_j) = h(x_k)$ and $x_j \ne x_k \Longrightarrow x_j = \dfrac{1}{x_k}$. Then jour constraint becomes

$$\begin{aligned} 1 &= \frac{1}{1+x_k} + \frac{1}{1+x_j} + (\text{other positive terms}) \\ &= \frac{1}{1+x_k} + \frac{1}{1+\dfrac{1}{x_k}} + (\text{other positive terms}) \\ &= \frac{1}{1+x_k} + \frac{x_k}{1+x_k} + (\text{other positive terms}) \\ &= 1 + (\text{other positive terms}) \end{aligned}$$

which is impossible. Therefore, $x_k = x_j$.

Hence, to achieve the extreme value, which must be a minimum, all of the x_i are equal and must equal $n-1$, forcing the minimum value of the product to be $(n-1)^n$.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Denote by $\dfrac{1}{1+x_i} = y_i \Longrightarrow x_i = \dfrac{1-y_i}{y_i},\ y_i > 0, i = 1, 2, , n$

By the AM-GM, we get

$$x_1x_2\ldots x_n = \prod_{i=1}^{n} \frac{1-y_i}{y_i} = \frac{y_2+y_3+\ldots+y_n}{y_1}\ldots\frac{y_1+y_2+\ldots+y_{n-1}}{y_n} \ge \frac{(n-1)^n\ \sqrt[n-1]{(y_1y_2\ldots y_n)^{n-1}}}{y_1y_2\cdots y_n} = (n-1)^n.$$

So, $x_1x_2\ldots x_n \ge (n-1)^n$. Equality occurs for $x_1 = x_2 = \ldots = x_n = n-1$.

Editor's comment : In addition to a general solution to this problem, the problem's author, **Arkady Alt of San Jose, CA,** also provided 4 different solutions for the cases $n = 2 = 3$.

Solution A.

Let $n = 3$. We have $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \dfrac{1}{1+x_3} = 1 \Longleftrightarrow$
$3 + 2(x_1+x_2+x_3) + x_1x_2 + x_2x_3 + x_3x_1 = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_2x_3 +$
$x_3x_1 + x_1x_2x_3 \iff 2 + x_1 + x_2 + x_3 = x_1x_2x_3$. Since $x_1 + x_2 + x_3 \ge 3\sqrt[3]{x_1x_2x_3}$

then $x_1x_2x_3 \geq 2+3\sqrt[3]{x_1x_2x_3} \iff \left(\sqrt[3]{x_1x_2x_3}-2\right)\left(\sqrt[3]{x_1x_2x_3}+1\right)^2 \geq 0 \iff$ $\sqrt[3]{x_1x_2x_3}-2 \geq 0 \iff x_1x_2x_3 \geq 2^3$.

Solution B.

Since $\dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}+\dfrac{1}{1+x_3}=1 \iff \dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}=\dfrac{x_3}{1+x_3} \iff$ $\dfrac{1+x_3}{1+x_1}+\dfrac{1+x_3}{1+x_2}=x_3 \implies x_3 \geq 2\left(1+x_3\right)\sqrt{\dfrac{1}{1+x_1}.\dfrac{1}{1+x_2}}=\dfrac{2\left(1+x_3\right)}{\sqrt{\left(1+x_1\right)\left(1+x_2\right)}}$.

Similarly we obtain $x_2 \geq \dfrac{2\left(1+x_2\right)}{\sqrt{\left(1+x_3\right)\left(1+x_1\right)}}, x_1 \geq \dfrac{2\left(1+x_1\right)}{\sqrt{\left(1+x_2\right)\left(1+x_3\right)}}$.

Hence, $x_1x_2x_3 \geq \dfrac{2^3\left(1+x_1\right)\left(1+x_2\right)\left(1+x_3\right)}{\sqrt{\left(1+x_2\right)\left(1+x_3\right)}\cdot\sqrt{\left(1+x_3\right)\left(1+x_1\right)}\cdot\sqrt{\left(1+x_1\right)\left(1+x_2\right)}}=2^3$.

Solution C.

Let $a:=\dfrac{1}{1+x_1}, b:=\dfrac{1}{1+x_2}, c:=\dfrac{1}{1+x_3}$ then $a,b,c \in (0,1), a+b+c=1$ and $x_1=\dfrac{1-a}{a}=\dfrac{b+c}{a} \geq \dfrac{2\sqrt{bc}}{a}, x_2=\dfrac{1-b}{b}=\dfrac{c+a}{b} \geq \dfrac{2\sqrt{ca}}{b}, x_3=\dfrac{1-c}{c}=\dfrac{a+b}{c} \geq \dfrac{2\sqrt{ab}}{c}$.

Therefore, $x_1x_2x_3 \geq \dfrac{2\sqrt{bc}}{a}\cdot\dfrac{2\sqrt{ca}}{b}\cdot\dfrac{2\sqrt{ab}}{c}=8$.

Solution D.

First note that at least one of the products x_1x_2, x_2x_3, x_3x_1 must be greater then 1.

Indeed,assume that $x_1x_2, x_2x_3, x_3x_1 \leq 1$. Then since $2+x_1+x_2+x_3=x_1x_2x_3 \iff$ $1=\dfrac{2}{x_1x_2x_3}+\dfrac{1}{x_1x_2}+\dfrac{1}{x_2x_3}+\dfrac{1}{x_3x_1}$ and $x_1x_2x_3=\sqrt{x_1x_2\cdot x_2x_3\cdot x_3x_1} \leq 1$

we obtain a contradiction $1=\dfrac{2}{x_1x_2x_3}+\dfrac{1}{x_1x_2}+\dfrac{1}{x_2x_3}+\dfrac{1}{x_3x_1} \geq 2+1+1+1 \geq 5$.

Let it be $x_1x_2>1$ and let $t:=\sqrt{x_1x_2}, r:=x_1x_2x_3$.

Then $2+x_1+x_2+x_3=x_1x_2x_3$ becomes

$+\dfrac{r}{t^2}=r$ and, since $x_1+x_2 \geq 2\sqrt{x_1x_2}=2t,\ t>1$, we obtain

$$r-\frac{r}{t^2}=2+x_1+x_2 \geq 2+2t \iff \frac{r\left(t^2-1\right)}{t^2} \geq 2\left(t+1\right) \iff r \geq \frac{2t^2}{t-1}=2\left(\frac{t^2-1+1}{t-1}\right)=$$

$$2\left(\left(t-1+\frac{1}{t-1}\right)+2\right) \geq 2\left(2+2\right)=8, \text{ because } t-1+\frac{1}{t-1} \geq 2.$$

Comment by Editor: **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru of Comănesti, Romania,** stated that there is a paper in the Romanian Mathematical Gazette, (Volume CXX, number 11, 2015) pp. 489-498 by **Eugen Păltănea** that presents five solutions and extensions for the following proposition: Let $x_1, x_2, \ldots, x_n>0, n \geq 2$. If $\dfrac{1}{1+x_1}+\dfrac{1}{1+x_2}+\ldots+\dfrac{1}{1+x_n}=1$, then $\sqrt[n]{x_1x_2\ldots x_n} \geq n-1$. They presented a new solution to this proposition and then applied it to problem 5404.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg,

Switzerland; and the authors.

- **5405:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a, b \in \Re$ such that $a + b = 1$, $e_n = \left(1 + \frac{1}{n}\right)^n$ and $c_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$, then compute

$$\lim_{n\to\infty} \left((n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{e_n} \right)^b.$$

Solution 1 by Ramya Dutta (student, Chennai Mathematical Institute) India

Using $\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$, for $-1 < x < 1$ and the Stirling Approximation:

$\log n! = \left(n + \frac{1}{2}\right) \log n - n + \frac{1}{2} \log 2\pi + O\left(\frac{1}{n}\right)$

For $n > 2$,

$$\begin{aligned}
(n!e_n)^{b/n} &= \exp\left(\frac{b \log n!}{n}\right) \left(1 + \frac{1}{n}\right)^b \\
&= \exp\left(b \log n + \frac{b \log n}{2n} - b + \frac{b \log 2\pi}{2n} + O\left(\frac{1}{n^2}\right)\right) \left(1 + \frac{1}{n}\right)^b \\
&= e^{-b} n^b \exp\left(\frac{b \log n}{2n} + \frac{b \log 2\pi}{2n} + O\left(\frac{1}{n^2}\right)\right) \left(1 + \frac{1}{n}\right)^b \\
&= e^{-b} n^b \left(1 + \frac{b \log n}{2n} + \frac{b \log 2\pi}{2n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \left(1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= e^{-b} n^b \left(1 + \frac{b \log n}{2n} + \frac{b \log 2\pi}{2n} + \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right)
\end{aligned}$$

Again, $c_n = H_n - \log n = \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$

Therefore,

$$\begin{aligned}
c_n^{b/(n+1)} &= \exp\left(\frac{b \log c_n}{n+1}\right) \\
&= \exp\left(\frac{b \log \gamma}{n+1} + \frac{b}{n+1} \log\left(1 + \frac{1}{2\gamma n} + O\left(\frac{1}{n^2}\right)\right)\right) \\
&= \exp\left(\frac{b \log \gamma}{n} + O\left(\frac{1}{n^2}\right)\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&((n+1)!c_n)^{b/(n+1)} \\
&= e^{-b}(n+1)^b \exp\left(\frac{b\log(n+1)}{2(n+1)} + \frac{b\log 2\pi}{2(n+1)} + O\left(\frac{1}{n^2}\right)\right) c_n^{b/(n+1)} \\
&= e^{-b}(n+1)^b \exp\left(\frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + O\left(\frac{1}{n^2}\right)\right) \exp\left(\frac{b\log\gamma}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= e^{-b}(n+1)^b \left(1 + \frac{b\log n}{2n} + \frac{b\log(2\pi\gamma^2)}{2n} + O\left(\frac{\log^2 n}{n^2}\right)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} (n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{(n!e_n)^b} \\
&= \lim_{n\to\infty} e^{-b}(n+1)\left(1 + \frac{b\log n}{2n} + \frac{b\log(2\pi\gamma^2)}{2n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&\quad - e^{-b}n\left(1 + \frac{b\log n}{2n} + \frac{b\log 2\pi}{2n} + \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&= \lim_{n\to\infty} e^{-b}\left(1 + O\left(\frac{\log n}{n}\right)\right) + e^{-b}n\left(\frac{b\log\gamma}{n} - \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&= \lim_{n\to\infty} e^{-b}(1 + b\log\gamma - b) + O\left(\frac{\log n}{n}\right) = e^{-b}(a + b\log\gamma)
\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The n^{th} harmonic number admits the asymptotic expansion $\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O\left(\frac{1}{n}\right)$, as $n \to \infty$. (See for instance *https://en/wikipedia.org/wiki/Harmonic_number.*)

Stirling's formula states that $n! = \sqrt{2\pi n}n^n e^{-n}\left(1 + O\left(\frac{1}{n}\right)\right)$, as $n \to \infty$. (See for instance *https://en/wikipedia.org/wiki/Stirling %27s approximation*).

So

$$\begin{aligned}
&(n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} = \\
&= (n+1)^{a+b}(2\pi)^{\frac{b}{2(n+1)}}(n+1)^{\frac{b}{2(n+1)}} e^{-b}\left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{b}{n+1}}\left(\gamma + O\left(\frac{1}{n}\right)\right)^{\frac{b}{n+1}} \\
&= (n+1)e^{-b}\left(1 + \frac{b}{2(n+1)}\log(2\pi) + \frac{b}{2(n+1)}\log(n+1) + \frac{b}{(n+1)}\log\gamma + O\left(\frac{\log^2 n}{n^2}\right)\right) \\
&= e^{-b}\left(n + 1 + \frac{b}{2}\log(2\pi) + \frac{b}{2}\log(n+1) + b\log\gamma + O\left(\frac{\log^2 n}{n}\right)\right),
\end{aligned}$$

$$n^n \sqrt[n]{(n!e_n)^b} = n^{a+b}(2\pi)^{\frac{b}{2n}} n^{\frac{b}{2n}} e^{-b} \left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{b}{n}} \left(1 + \frac{1}{n}\right)^b$$

$$= \quad ne^{-b}\left(1 + \frac{b}{2n}\log(2\pi) + \frac{b}{2n}\log(n) + \frac{b}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right),$$

$$= \quad e^{-b}\left(n + \frac{b}{2}\log(2\pi) + \frac{b}{2}\log(n) + b + O\left(\frac{\log^2 n}{n}\right)\right).$$

Thus

$$\lim_{n\to\infty}\left((n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{(n!e_n)^b}\right)$$

$$= \lim_{n\to\infty}\left(e^{-b}\left(n + 1 + \frac{b}{2}\log(2\pi) + \frac{b}{2}\log(n+1) + b\log(\gamma) - n - \frac{b}{2}\log(2\pi) - \frac{b}{2}\log(n) - b + O\left(\frac{\log^2 n}{n}\right)\right)\right)$$

$$= \quad e^{-b}\left(1 + b\log\gamma - b\right) = e^{-b}\left(a + b\log\gamma\right).$$

Also solved by Arkady Alt, San Jose, CA; Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the proposers.

- **5406:** *Proposed by Cornel Ioan Vălean, Timis, Romania*

Calculate:

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the harmonic number.

Solutions 1 and 2 by Ramya Dutta (student), Chennai Mathematical Institute India

Solution (1):

Changing the order of summation in $(\star)$ and using $\sum_{n=1}^{k} \frac{H_n}{n} = \frac{H_k^2 + H_k^{(2)}}{2}$, we have:

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(3) - \sum_{k=1}^{n}\frac{1}{k^3}\right) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{H_n}{nk^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^4} \qquad (\star)$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{k}\frac{H_n}{n} - \sum_{n=1}^{\infty}\frac{H_n}{n^4}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k^2 + H_k^{(2)}}{k^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^4}$$

Lemma: $\displaystyle\sum_{k=1}^{\infty}\frac{H_k}{k(n+k)}=\frac{1}{n}\left(\frac{1}{2}H_n^2+\frac{1}{2}H_n^{(2)}+\zeta(2)-\frac{H_n}{n}\right)$

Proof:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{H_k}{k(n+k)} &= \sum_{k=1}^{\infty}\sum_{j=1}^{k}\frac{1}{jk(n+k)}=\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\frac{1}{jk(n+k)} && (1)\\
&= \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\frac{1}{jk(n+k)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (2)\\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(k+j)(n+k+j)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (3)\\
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{jk(n+k+j)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (4)\\
&= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{n+k}}{k(n+k)}+\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)} && (5)\\
&= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{n+k}}{k(n+k)}+\frac{1}{n}\left(\zeta(2)-\frac{H_n}{n}\right) && (6)
\end{aligned}
$$

Justifications: (1) Interchanged order of summation, (3) made the change in variable $k\mapsto k+j$, (4) used the symmetry of the summation w.r.t. k and j,

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(k+j)(n+k+j)}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(k+j)(n+k+j)}\\
&=\frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(k+j)(n+k+j)}+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(k+j)(n+k+j)}\right)\\
&=\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{jk(n+k+j)},
\end{aligned}
$$

(5) used the identity, $\displaystyle\frac{H_m}{m}=\frac{1}{m}\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{m+j}\right)=\sum_{j=1}^{\infty}\frac{1}{j(m+j)}$ and

(6) used partial fraction, $\displaystyle\sum_{j=1}^{\infty}\frac{1}{j^2(n+j)}=\sum_{j=1}^{\infty}\left(\frac{1}{nj^2}-\frac{1}{nj(n+j)}\right)=\frac{1}{n}\left(\zeta(2)-\frac{H_n}{n}\right).$

Again,

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{H_{n+k}}{k(n+k)} &= \frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{H_k}{k}-\frac{H_{n+k}}{n+k}\right)+\frac{1}{n}\sum_{k=1}^{\infty}\left(\frac{H_{n+k}-H_k}{k}\right) &(7)\\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k}+\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{j=1}^{n}\frac{1}{k+j}\right) &(8)\\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+j}\right) &(9)\\
&= \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k}=\frac{H_n^2+H_n^{(2)}}{n} &(10)
\end{aligned}$$

Thus, combining lines (6) and (10),

$$\sum_{k=1}^{\infty}\frac{H_k}{k(n+k)}=\frac{1}{n}\left(\frac{1}{2}H_n^2+\frac{1}{2}H_n^{(2)}+\zeta(2)-\frac{H_n}{n}\right) \qquad \square$$

Now, dividing both sides of the identity with n^2 and summing over $n \geq 1$,

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n^3}+\zeta(2)\zeta(3)-\sum_{n=1}^{\infty}\frac{H_n}{n^4} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k}{kn^2(n+k)}\\
&= \sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\zeta(2)-\frac{H_k}{k}\right)
\end{aligned}$$

where, we used partial fraction decomposition from line (6) earlier. That is,

$$\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n^3}=\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\zeta(2)\zeta(3)+\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} \qquad (\mathbf{I})$$

Now we provide an evaluation of the Euler sum: $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}$.

Consider the partial fraction decomposition,

$$\begin{aligned}
\sum_{k=1}^{n-1}\left(\frac{1}{k(n-k)}\right)^2 &= \frac{1}{n^2}\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right)^2\\
&= \frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{k^2}+\frac{1}{(n-k)^2}+\frac{2}{n}\left(\frac{1}{k}+\frac{1}{n-k}\right)\\
&= \frac{2}{n^2}\left(H_n^{(2)}+\frac{2H_n}{n}-\frac{3}{n^2}\right)
\end{aligned}$$

Dividing both sides by n and summing over $n \geq 1$,

$$
\begin{aligned}
2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(H_n^{(2)}+\frac{2H_n}{n}-\frac{3}{n^2}\right) &= \sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{1}{nk^2(n-k)^2} \qquad \text{(change of variable } n=m+k\text{)}\\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2m^2(k+m)}\\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{k(m+k)-k^2}{k^3m^3(k+m)}\\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2m^3}-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{km^3(k+m)}\\
&= \zeta(2)\zeta(3)-\sum_{m=1}^{\infty}\frac{H_m}{m^4}
\end{aligned}
$$

i.e.,

$$
\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=\frac{1}{2}\zeta(2)\zeta(3)-\frac{5}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+3\zeta(5) \qquad \textbf{(II)}
$$

Thus, combining the results from (**I**) and (**II**),

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right) &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n^3}-\sum_{n=1}^{\infty}\frac{H_n}{n^4}\\
&= \frac{1}{3}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\frac{1}{3}\zeta(2)\zeta(3)-\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}\\
&= \frac{1}{3}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\frac{1}{6}\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\zeta(5)
\end{aligned}
$$

Using Euler's summation formula:

$$
\sum_{n=1}^{\infty}\frac{H_n}{n^q}=\left(1+\frac{q}{2}\right)\zeta(q+1)-\frac{1}{2}\sum_{j=1}^{q-2}\zeta(j+1)\zeta(q-j), \qquad \text{for } q\geq 2
$$

we have the particular cases, $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3)$ and $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^4}=3\zeta(5)-\zeta(2)\zeta(3)$,

i.e.,

$$
\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right)=2\zeta(2)\zeta(3)-\frac{7}{2}\zeta(5)
$$

Solution (2):

We start with evaluating the integral for $a > 0$,

$$\begin{aligned}\int_0^1 x^{a-1}\log^2(1-x)\,dx &= \lim_{b\to 1}\frac{\partial^2}{\partial b^2}\int_0^1 x^{a-1}(1-x)^{b-1}\,dx\\ &= \lim_{b\to 1}\frac{\partial^2}{\partial b^2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\\ &= \frac{1}{a}\Big((\gamma+\psi(a+1))^2+\zeta(2)-\psi^{(1)}(a+1)\Big)\end{aligned}$$

Thus, $\displaystyle\int_0^1 x^{n-1}\log^2(1-x)\,dx = \frac{H_n^2+H_n^{(2)}}{n}$

So,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right) &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n^3}-\sum_{n=1}^{\infty}\frac{H_n}{n^4}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1}\log^2(1-x)\,dx-\sum_{n=1}^{\infty}\frac{H_n}{n^4}\\ &= \frac{1}{2}\int_0^1\frac{\mathrm{Li}_2(x)\log^2(1-x)}{x}\,dx-\sum_{n=1}^{\infty}\frac{H_n}{n^4}\end{aligned}$$

Using the reflection formula for Dilogarithm,
$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-\log x\log(1-x)$
we may rewrite the integral as,

$$\begin{aligned}&\int_0^1\frac{\mathrm{Li}_2(x)\log^2(1-x)}{x}\,dx\\ &=\zeta(2)\underbrace{\int_0^1\frac{\log^2(1-x)}{x}\,dx}_{\textbf{(I)}}-\underbrace{\int_0^1\frac{\log x\log^3(1-x)}{x}\,dx}_{\textbf{(II)}}-\underbrace{\int_0^1\frac{\mathrm{Li}_2(1-x)\log^2(1-x)}{x}\,dx}_{\textbf{(III)}}\end{aligned}$$

The first integral **(I)**:

$$\begin{aligned}\int_0^1\frac{\log^2(1-x)}{x}\,dx &= \int_0^1\frac{\log^2 x}{1-x}\,dx\\ &= \sum_{n=1}^{\infty}\int_0^1 x^{n-1}\log^2 x\,dx\\ &= 2\sum_{n=1}^{\infty}\frac{1}{n^3}=2\zeta(3)\end{aligned}$$

The second integral **(II)**: Using $\displaystyle\frac{\log(1-x)}{1-x}=-\sum_{n=1}^{\infty}H_nx^n$,

$$\begin{aligned}\int_0^1\frac{\log x\log^3(1-x)}{x}\,dx &= \int_0^1\frac{\log^3 x\log(1-x)}{1-x}\,dx\\ &= -\sum_{n=1}^{\infty}\int_0^1 H_nx^n\log^3 x\,dx\\ &= 6\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{H_n}{n^4}-6\zeta(5)\end{aligned}$$

The third integral **(III)**:

$$\int_0^1 \frac{\mathrm{Li}_2(1-x)\log^2(1-x)}{x}\,dx = \int_0^1 \frac{\mathrm{Li}_2(x)\log^2 x}{1-x}\,dx$$
$$= \sum_{n=1}^{\infty}\int_0^1 H_n^{(2)} x^n \log^2 x\,dx$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3} = 2\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} - 2\zeta(5)$$

Combining the results,

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-\sum_{k=1}^{n}\frac{1}{k^3}\right) = \zeta(2)\zeta(3) - 4\sum_{n=1}^{\infty}\frac{H_n}{n^4} + 4\zeta(5) - \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}$$
$$= \frac{1}{2}\zeta(2)\zeta(3) - \frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4} + \zeta(5)$$
$$= 2\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5)$$

Editor′s comment : **Albert Stadler of Herrliberg, Switzerland** mentioned in his solution that the expression $\sum_{k=1}^{\infty}\frac{H_k}{k^4} = -\frac{\pi^2}{6}\zeta(3) + 3\zeta(5)$ is due to Euler and that Euler went on to generalize this formula as follows:

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^m} = m + 2\zeta(m+1) - \sum_{n=1}^{m-2}\zeta(m-n)\zeta(n+1), m = 2, 3, \ldots$$

The reference he gave for this is: L.Euler, Meditationes circa singulare serierum genus, Novi Comm. Acad. Sci. Petropolitanae 20 (1775), 140-186. Reprinted in Opera Omnia, ser. I, vol. 15, B.G. Teubner, Berlin, 1927, pp 217-267.

Solution 3 by Moti Levy, Rehovot, Israel

We calculate the sum by expressing it as a sum of definite integrals (involving polylogarithmic function) and then make use of results by Prof. Pedro Freitas [1].
The tail of $\zeta(3)$ is

$$\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} = \sum_{k=1}^{\infty}\frac{1}{(n+k)^3}. \qquad (11)$$

The following definite integral is known [2]:

$$\int_0^1 x^n \ln^2 x dx = \frac{2}{(n+1)^3}. \qquad (12)$$

Substituting (11) in (12) and changing the order of summation and integration give,

$$\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} = \frac{1}{2}\sum_{k=1}^{\infty}\int_0^1 x^{n+k-1}\ln^2 x dx$$
$$= \frac{1}{2}\int_0^1 x^n \ln^2 x \sum_{k=1}^{\infty} x^{k-1} dx = \frac{1}{2}\int_0^1 \frac{x^n}{1-x}\ln^2 x dx.$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)=\sum_{n=1}^{\infty}\frac{H_n}{n}\frac{1}{2}\int_0^1\frac{x^n}{1-x}\ln^2 xdx=\frac{1}{2}\int_0^1\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\frac{\ln^2 x}{1-x}dx \quad (13)$$

Let $F(x):=\sum_{n=1}^{\infty}\frac{H_n}{n}x^n$, then $\frac{dF}{dx}=\frac{1}{x}\sum_{n=0}^{\infty}H_nx^n$. The generating function of the sequence $(H_n)_{n\geq 0}$ is well known [3]

$$\sum_{n=0}^{\infty}H_nx^n=-\frac{\ln(1-x)}{1-x}.$$

It follows that $\frac{dF}{dx}=-\frac{\ln(1-x)}{x(1-x)}$. To find $F(x)$ we integrate,

$$F(x)=-\int_0^x\frac{\ln(1-t)}{t(1-t)}dt=-\int_0^x\frac{\ln(1-t)}{1-t}dt-\int_0^x\frac{\ln(1-t)}{t}dt=\frac{1}{2}\ln^2(1-x)+\mathrm{Li}_2(x) \quad (14)$$

Now we substitute (14) in (13) and obtain the required sum as a sum of two definite integrals,

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)=\frac{1}{4}\int_0^1\frac{\ln^2 x\ln^2(1-x)}{1-x}dx+\frac{1}{2}\int_0^1\frac{\ln^2 x}{1-x}\mathrm{Li}_2(x)\,dx.$$

These definite integrals appear in [1] as entries in Table 6:

$$\int_0^1\frac{\ln^2 x\ln^2(1-x)}{1-x}dx=-4\zeta(2)\zeta(3)+8\zeta(5).$$

$$\int_0^1\frac{\ln^2 x}{1-x}\mathrm{Li}_2(x)\,dx=6\zeta(2)\zeta(3)-11\zeta(5).$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)&=\frac{1}{2}(6\zeta(2)\zeta(3)-11\zeta(5))+\frac{1}{4}(-4\zeta(2)\zeta(3)+8\zeta(5))\\&=2\zeta(2)\zeta(3)-\frac{7}{2}\zeta(5)=\frac{\pi^2}{3}\zeta(3)-\frac{7}{2}\zeta(5)\cong 0.325\,36.\end{aligned}$$

References:
[1] Freitas Pedro, "*Integrals of Polylogarithmic functions, recurrence relations, and associated Euler sums*", arXiv:math/0406401v1 [math.CA] 21 Jun 2004.
[2] Gradshteyn and Ryzhik, "*Table of Integrals, Series and Products*" (7Ed, Elsevier, 2007), Entry **2.723-2**.
[3] Ronald L. Graham, Donald E. Knuth, Oren Patashnik "*Concrete Mathematics, A Foundation for Computer Science*", 2nd Edition 1994, page 352, (7.43).

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the sum of the problem, denoted by S, equals $\dfrac{4\zeta(2)\zeta(3)-7\zeta(5)}{2}$.

We need the facts that

$$\frac{H_n}{n}=-\int_0^1 x^{n-1}\ln(1-x)dx, \quad \text{(see p. 206, of [2])},$$

$$\frac{1}{(n+m)^3}=\frac{1}{2}\int_0^1 x^{m+n-1}\ln^2 xdx, \quad \text{(see formula 2.723 of [3]), and}$$

$$\gamma(3)-\sum_{m=1}^{n}\frac{1}{m^3} = \sum_{m=1}^{\infty}\frac{1}{(n+m)^3}.$$

For $-1\leq 1$ let $\text{Li}_2(x)=\sum_{m=1}^{\infty}\frac{x^n}{n^2}$. By following closely the method of solution of problem 3.62 in [2, $p.$ 211 – 213], we obtain,

$$\begin{aligned} S=-\frac{1}{2}\int_0^1\int_0^1\frac{y\ln^2 y\ln(1-x)}{(1-y)(1-xy)}dxdy &= -\frac{1}{2}\int_0^1\frac{y\ln^2 y}{1-y}\left(\frac{-\frac{1}{2}\ln^2(1-y)-Li_2(y)}{y}\right)dy \\ &= \frac{1}{4}I+\frac{1}{2}J, \end{aligned}$$

where $I=\int_0^1\frac{\ln^2 y\ln^2(1-y)}{1-y}dy$ and $J=\int_0^1\frac{\ln^2 y_2(1-y)}{1-y}dy$. It is known [1, p.1436, Table 6] that $I=8\zeta(5)-4\zeta(2)\zeta(3)$ and $J=6\zeta(2)\zeta(3)-11\zeta(5)$.

Hence the claimed result for the sum of the problem.

References:
1. Freitas P.: Integrals of polylogarithmic functions, recurrence relations and associated Euler sums, Mathematics of Computation, vol. 74, number 251, 1425-1440 (2005).

2. Furdui O.: Limits, Series, and Fractional Part Integrals, Springer, New Hork, (2013)

3. Gradshteyn, I.S. and Ryzhik, I.M.: Tables of Intgerals, Series, and Products, Seventh Edition, Elsevier (2007).

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5407:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all triples (a,b,c) of positive reals such that

$$\begin{aligned} a+b+c &= 1, \\ \frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} &= \frac{243}{16}. \end{aligned}$$

Solution 1 by Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru of Comănesti, Romania

Since $a+b+c=1$ then $a+bc=a\cdot 1+bc=a(a+b+c)+bc=(a+b)(a+c)$. We denote $a+b=x, b+c=y$ and $c+a=z$ then $x+y+z=2$. Using well-known inequalities we have

$$\frac{243}{16} = \frac{1}{x^2y^2}+\frac{1}{y^2z^2}+\frac{1}{z^2x^2}$$

$$
\begin{aligned}
&\geq \frac{1}{xy}\cdot\frac{1}{yz}+\frac{1}{yz}\cdot\frac{1}{zx}+\frac{1}{zx}\cdot\frac{1}{xy}\\
&= \frac{1}{xyz}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq \frac{1}{\frac{x+y+z}{3}}\cdot\frac{9}{x+y+z}\\
&= \frac{27}{8}\cdot\frac{9}{2}=\frac{243}{16}.
\end{aligned}
$$

Hence, $x = y = z \Longrightarrow a = b = c = \dfrac{1}{3}$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Assume that $a, b, c > 0$ and $a + b + c = 1$. Then, by the Arithmetic - Geometric Mean Inequality,

$$a + bc \leq a + \frac{(b+c)^2}{4} = a + \frac{(1-a)^2}{4} = \frac{(a+1)^2}{4},$$

with equality if and only if $b = c$. Since $a, b, c > 0$, it follows that

$$\frac{1}{(a+bc)^2} \geq \frac{16}{(a+1)^4}, \tag{1}$$

with equality if and only if $b = c$. Similar steps show that

$$\frac{1}{(b+ca)^2} \geq \frac{16}{(b+1)^4}, \tag{2}$$

with equality if and only if $c = a$, and

$$\frac{1}{(c+ab)^2} \geq \frac{16}{(c+1)^4}, \tag{3}$$

with equality if and only if $a = b$. By (1), (2), and (3), we have

$$\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} \geq 16\left[\frac{1}{(a+1)^4}+\frac{1}{(b+1)^4}+\frac{1}{(c+1)^4}\right], \tag{4}$$

with equality if and only if $a = b = c = \dfrac{1}{3}$.

Further, if $f(x) = \dfrac{1}{x^4}$, then $f''(x) = \dfrac{20}{x^6} > 0$ on $(0, \infty)$, and hence, $f(x)$ is strictly convex on $(0, \infty)$. If we use Jensen's Theorem, we obtain

$$
\begin{aligned}
\frac{1}{(a+1)^4}+\frac{1}{(b+1)^4}+\frac{1}{(c+1)^4} &= f(a+1)+f(b+1)+f(c+1)\\
&\geq 3f\left[\frac{(a+1)+(b+1)+(c+1)}{3}\right]\\
&= 3f\left(\frac{4}{3}\right)\\
&= \frac{243}{256}, \qquad (5)
\end{aligned}
$$

with equality if and only if $(a+1)=(b+1)=(c+1)$, i.e., if and only if $a=b=c=\dfrac{1}{3}$.
By combining (4) and (5), we see that the conditions $a,b,c>0$ and $a+b+c=1$ imply that

$$\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2}\geq 16\left(\frac{243}{256}\right)=\frac{243}{16},$$

with equality if and only if $a=b=c=\dfrac{1}{3}$. Therefore, the unique solution for our system must be $a=b=c=\dfrac{1}{3}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$(1-a)^2-4bc=(b+c)^2-4bc=(b-c)^2\geq 0$ with equality iff $b=c$

$\Longrightarrow \dfrac{(1+a)^2}{4}=a+\dfrac{(1-a)^2}{4}\geq a+bc>0$ with equality iff $b=c \Longrightarrow \dfrac{1}{(a+bc)^2}\geq\dfrac{16}{(1+a)^4}$ with equality iff $b=c$, and cyclically, so

$$\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2}\geq 16\left(\frac{1}{(1+a)^4}+\frac{1}{(1+b)^4}+\frac{1}{(1+c)^4}\right)$$

with equality iff $a=b=c=\dfrac{1}{3}$. By the arithmetic mean−geometric mean inequality,

$$\begin{aligned}\frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} &\geq 16\cdot 3\sqrt[3]{\frac{1}{(1+a)^4(1+b)^4(1+c)^4}}\\ &= \frac{48}{\left(\sqrt[3]{(1+a)(1+b)(1+c)}\right)^4}\\ &\geq \frac{48}{\left(\dfrac{1+a+1+b+1+c}{3}\right)^4}=\frac{48}{\left(\dfrac{4}{3}\right)^4}=\frac{243}{16}\end{aligned}$$

with equality iff $a=b=c=\dfrac{1}{3}$, so from this and the second of the given equations we conclude that $a=b=c=\dfrac{1}{3}$.

Editor′s comment: **D.M. Bătinetu-Giurgiu, of "Matei Basarab" National College, Bucharest, Romanina and Neculai Stanciu of "George Emil Palade" School Buzău, Romania** generalized the problem as follows:

If $a,b\geq 0, a+b,c,d,m>0,x,y,z>0$, such that $x+y+z=s>0$ and

$$\frac{(as+bx)^{m+1}}{(cx+dyz)^m}+\frac{(as+bt)^{m+1}}{(cy+dzx)^m}+\frac{(as+bz)^{m+1}}{(cz+dxy)^m}=\frac{3^m(3a+b)^{m+1}s}{(3c+ds)^m},\text{then find all triples}(x,y,z).$$

They found the solution that since $x+y+z=s$, then $s^2\geq 3(xy+yz+zx)$ with equality iff $x=y=z=\dfrac{s}{3}$.

If $s=1, m=2, a=1, b=0, c=1, d=1$ we obtain $x+y+z=1$ and $\sum_{cyc} \dfrac{1}{(x+yz)^2} = \dfrac{243}{16}$, i.e., problem 5407.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Ramya Dutta (student), Chennai Mathematical Institute, India; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro,GA, and the proposer.

- **5408:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate:

$$\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)} dx.$$

Solution 1 by Albert Stadler, Herriliberg, Switzerland

$$\begin{aligned}
\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}\, dx &= \int_0^1 \left(\frac{1}{x} - \frac{1}{1-x}\right) \ln(x)\ln(1-x) dx \\
&= 2\int_0^1 \frac{(\ln x)(\ln(1-x))}{x} dx \\
&= -2\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln x dx \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3).
\end{aligned}$$

Solution 2 by Moti Levy, Rehovot, Israel

Since $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$ and $\int_0^1 \frac{\ln x \ln(1-x)}{x} dx = \int_0^1 \frac{\ln x \ln(1-x)}{(1-x)} dx$ then

$$I := \int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)} dx = 2\int_0^1 \frac{\ln x \ln(1-x)}{x} dx.$$

Using the Taylor series of $\ln(1-x)$ for $0 < x < 1$, and changing the order of summation and integration,

$$I = -2\int_0^1 \frac{\ln x}{x} \left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right) dx = -2\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^{k-1} \ln x dx.$$

Gradshteyn and Ryzhik, entry **2.723-1,**

$$\int x^n \ln x dx = x^{n+1} \left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right).$$

$$I = -2\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^{k-1} \ln x dx = 2\sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{k^2} = 2\zeta(3).$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the integral of the problem , denoted by I equals $2\sum_{n=1}^{\infty}\frac{1}{n^3}$.

It is well known that for non-negative integers n.

$$\int x^n \ln x dx = x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right) + constant.$$

Hence for $0 < a < 1$, we have

$$\begin{aligned}\int_0^a \frac{\ln x\ (1-x)}{x}dx &= -\int_0^a \ln x \sum_{n=0}^{\infty}\frac{x^n}{n+1}dx = -\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^a x^n \ln x dx \\ &= -\ln a \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)^2} + \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)^3},\ \text{so that} \\ \int_0^1 \frac{\ln x\ (1-x)}{x}dx &= \sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)^3}.\end{aligned}$$

Since $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$, so

$$I = \int_0^1 \frac{\ln x \ln(1-x)}{x}dx + \int_0^1 \frac{\ln x \ln(1-x)}{1-x}dx = 2\int_0^1 \frac{\ln x \ln(1-x)}{x}dx = 2\sum_{n=0}^{\infty}\frac{1}{(n+1)^3},$$

as asserted.

Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA

A generating function for the Harmonic numbers is

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}.$$

The radius of convergence for this series is 1, so the order of summation and integration can be reversed to yield

$$\begin{aligned}\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}dx &= -\int_0^1 \frac{\ln x}{x}\left(\sum_{n=1}^{\infty} H_n x^n\right)dx \\ &= -\sum_{n=1}^{\infty} H_n \int_0^1 x^{n-1}\ln x dx \\ &= \sum_{n=1}^{\infty}\frac{H_n}{n^2}\end{aligned}$$

$$= \quad 2\zeta(3).$$

Solution 5 by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

Let I denote the above integral and let $f(x) = \ln x \ln(1-x)$ and $g'(x) = \dfrac{1}{x(1-x)} = \dfrac{1}{x} + \dfrac{1}{1-x}$. Then $f'(x) = \dfrac{\ln(1-x)}{x} - \dfrac{\ln x}{1-x}$ and $g(x) = \ln x - \ln(1-x)$. Evaluating I by parts we have

$$\begin{aligned}
I &= [f(x)g(x)]_0^1 - \int_0^1 f'(x)g(x)dx \\
&= [\ln x \ln(1-x)(\ln x - \ln(1-x))]_0^1 - \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln x}{1-x}\right)(\ln x - \ln(1-x))dx \\
&= \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln x}{1-x}\right)(\ln(1-x) - \ln x)dx \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}dx + \int_0^1 \frac{\ln^2 x}{1-x}dx - \int_0^1 \frac{\ln x \ln(1-x)}{1-x}dx - \int_0^1 \frac{\ln x \ln(1-x)}{x}dx
\end{aligned}$$

Let $I_1 = \int_0^1 \dfrac{\ln^2(1-x)}{x}dx$, then $\int_0^1 \dfrac{\ln^2 x}{1-x}dx = I_1$ (with the substitution $y = 1-x$). Similarly let $I_2 = \int_0^1 \dfrac{\ln x \ln(1-x)}{x}dx$, then $\int_0^1 \dfrac{\ln x \ln(1-x)}{1-x}dx = I_2$ (with the substitution $y = 1-x$). So, $I = 2(I_1 - I_2)$. But we also notice that integration of I_2 by parts yields (taking $1/x$ as second function)

$$\begin{aligned}
I_2 = \int_0^1 \frac{\ln x \ln(1-x)}{x}dx &= \left[\ln^2 x \ln(1-x)\right]_0^1 - \int_0^1 \left(\frac{\ln(1-x)}{x} - \frac{\ln x}{1-x}\right)\ln x dx \\
&= \int_0^1 \frac{\ln^2 x}{1-x}dx - \int_0^1 \frac{\ln x \ln(1-x)}{x}dx = I_1 - I_2.
\end{aligned}$$

Thus $I_2 = \frac{1}{2}I_1$ and so $I = 2(I_1 - I_2) = I_1$. Now to calculate I_1, we notice that

$$I_1 = \int_0^1 \frac{\ln^2 x}{1-x}dx = \sum_{n=0}^{\infty} \int_0^1 x^n \ln^2 x dx \tag{15}$$

Now from integration by parts we have (by taking x^n as the second function)

$$\begin{aligned}
\int_0^1 x^n \ln^2 x dx &= \left[(\ln^2 x)\frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{2\ln x}{x} \cdot \frac{x^{n+1}}{n+1}dx = -\frac{2}{n+1}\int_0^1 x^n \ln x dx \\
&= -\frac{2}{n+1}\left[\left[(\ln x)\frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{x} \cdot \frac{x^{n+1}}{n+1}dx\right] = \frac{2}{n+1}\int_0^1 \frac{x^n}{n+1}dx = \frac{2}{(n+1)^3}.
\end{aligned}$$

Substituting the result obtained above in (1), we get $I_1 = \sum_{n=0}^{\infty} \dfrac{2}{(n+1)^3} = 2\zeta(3)$. Thus, $I = I_1 = 2\zeta(3)$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Pat Costello, Eastern Kentucky University, Richmond, KY; Ramya Dutta (student Chennai Mathematical Institute), India; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

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Henry Ricardo of the New York Math Circle should have been credited for having solved problem 5397.