Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

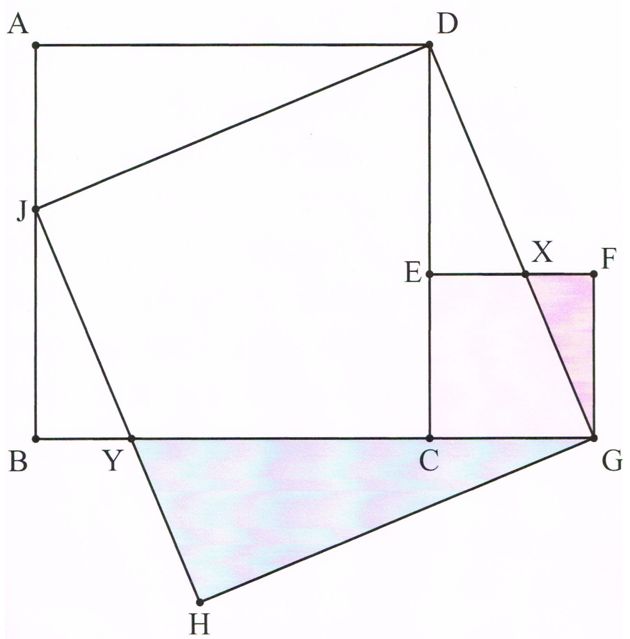
Solutions to the problems stated in this issue should be posted before June 15, 2017

- **5445:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of a triangle with exradii $(3, 4, 5)$.

- **5446:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Polygons $ABCD, CEFG$, and $DGHJ$ are squares. Moreover, point E is on side $DC, X = DG \cap EF$, and $Y = BC \cap JH$. If GX splits square $CEFG$ in regions whose areas are in the ratio 5:19. What part of square $DGHJ$ is shaded? (Shaded region in $DGHJ$ is composed of the areas of triangle YHG and trapezoid $EXGC$.)



- **5447:** *Proposed by Iuliana Trască, Scornicesti, Romanai*

  Show that if x, y, and z is each a positive real number, then
  $$\frac{x^6\cdot z^3+y^6\cdot x^3+z^6\cdot y^3}{x^2\cdot y^2\cdot z^2} \geq \frac{x^3+y^3+z^3+3x\cdot y\cdot z}{2}.$$

- **5448:** *Proposed by Yubal Barrios and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

  Evaluate: $\lim\limits_{n\to\infty}\sqrt[n]{\sum\limits_{\substack{0\leq i,j\leq n\\ i+j=n}}\binom{2i}{i}\binom{2j}{j}}.$

- **5449:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the use of a computer, find the real roots of the equation
  $$x^6-26x^3+55x^2-39x+10=(3x-2)\sqrt{3x-2}.$$

- **5450:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let k be a positive integer. Calculate
  $$\int_0^1\int_0^1\left\lfloor\frac{x}{y}\right\rfloor^k\frac{y^k}{x^k}\,dxdy,$$
  where $\lfloor a\rfloor$ denotes the floor (the integer part) of a.

Solutions

5427: *Proposed by Kenneth Korbin, New York, NY*

Rationalize and simplify the fraction

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)}\quad\text{if}\quad x=\frac{2017+\sqrt{2017-\sqrt{2017}}}{2017-\sqrt{2017-\sqrt{2017}}}.$$

Solution 1 by David E. Manes, SUNY at Oneonta, Oneonta, NY

Let $F=(x+1)^4/(x(2016x^2-2x+2016))$ and let $y=\sqrt{2017-\sqrt{2017}}$. Then $y^2=2017-\sqrt{2017}$ and $y^4=2017(2018-2\sqrt{2017})$. Moreover,

$$x=\frac{2017+y}{2017-y},\quad \frac{1}{x}=\frac{2017-y}{2017+y},\quad x+1=\frac{2(2017)}{2017-y}\quad\text{and}$$

$$x^2+1=\Big(\frac{2017+y}{2017-y}\Big)^2+1=\frac{(2017+y)^2+(2017-y)^2}{(2017-y)^2}$$
$$=\frac{2(2017^2+y^2)}{(2017-y)^2}.$$

Therefore,

$$2016(x^2+1)-2x=2\Big[\frac{2016(2017^2+y^2)}{(2017-y)^2}-\frac{2017+y}{2017-y}\Big]$$
$$=2\Big[\frac{2016(2017^2+y^2)-(2017^2-y^2)}{(2017-y)^2}\Big]$$
$$=2\Big[\frac{2015\cdot 2017^2+2017y^2}{(2017-y)^2}\Big]$$
$$=2(2017)\Big[\frac{2015(2017)+y^2}{(2017-y)^2}\Big]$$

Substituting these values into the fraction F and simplifying, we obtain

$$F=\frac{\left(\frac{2(2017)}{2017-y}\right)^4(2017-y)}{(2017+y)(2(2017)\left(\frac{2015(2017)+y^2}{(2017-y)^2}\right)}$$

$$=\frac{(2(2017))^3}{(2017^2-y^2)(2015\cdot 2017+y^2)}$$
$$=\frac{8(2017)^3}{2015\cdot 2017^3+2\cdot 2017(2017-\sqrt{2017})-2017(2018-2\sqrt{2017})}$$

$$=\frac{8(2017)^2}{2015\cdot 2017^2+2016}$$

$$=\frac{32546312}{8197604351}$$

$$\approx 0.003\,970\,222\,349.$$

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

For notational convenience we set $d=2017-\sqrt{2017}$, $y=2017+\sqrt{d}$, and $z=2017-\sqrt{d}$. Thus our x is y/z. We have

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{\left(\frac{y}{z}+1\right)^4}{\left(\frac{y}{z}\right)\left(2016\left(\frac{y}{z}\right)^2-2\left(\frac{y}{z}\right)+2016\right)}\cdot\frac{z^4}{z^4}$$
$$= \frac{(y+z)^4}{yz(2016y^2-2yz+2016z^2)}$$

Now

$$y+z=2\cdot 2017,$$

$$\begin{aligned} yz &= 2017^2 - d \\ &= 2017^2 - 2017 + \sqrt{2017} \\ &= 2017 \cdot 2016 + \sqrt{2017}, \end{aligned}$$

and

$$\begin{aligned} 2016y^2 - 2yz + 2016z^2 &= 2016(y^2 + z^2) - 2yz \\ &= 2016((y+z)^2 - 2yz) - 2yz \\ &= 2016(y+z)^2 - 2 \cdot 2017yz \\ &= 2016(2 \cdot 2017)^2 - 2 \cdot 2017(2017 \cdot 2016 + \sqrt{2017}) \\ &= 2 \cdot 2017(2017 \cdot 2016 - \sqrt{2017}). \end{aligned}$$

Hence

$$\begin{aligned} \frac{(y+z)^4}{yz(2016y^2 - 2yz + 2016z^2)} &= \frac{2^4 \cdot 2017^4}{2 \cdot 2017(2017^2 \cdot 2016^2 - 2017)} \\ &= \frac{2^3 \cdot 2017^2}{2017 \cdot 2016^2 - 1} \\ &= \frac{32546312}{8197604351}. \end{aligned}$$

Solution 3 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Let $y = 2017$ and $w = \sqrt{y - \sqrt{y}}$. Observe

$$\begin{aligned} x &= \frac{y+w}{y-w} \\ x+1 &= \frac{2y}{y-w} \\ w^2 &= y - \sqrt{y} \\ w^4 &= y^2 + y - 2y\sqrt{y}. \end{aligned}$$

Then

$$\begin{aligned} \frac{(x+1)^4}{x(2016x^2 - 2x + 2016)} &= \frac{2^4y^4}{(y-w)^4} \cdot \frac{y-w}{y+w} \cdot \frac{1}{2016\left(\frac{y+w}{y-w}\right)^2 - 2\left(\frac{y+w}{y-w}\right) + 2016} \\ &= \frac{2^4y^4}{2016(y+w)^3(y-w) - 2(y+w)^2(y-w)^2 + 2016(y+w)(y-w)^3} \\ &= \frac{2^4y^4}{2(2015y^4 + 2y^2w^2 - 2017w^4)} \\ &= \frac{2^3y^3}{2015y^3 + 2yw^2 - w^4} \qquad \text{using } y = 2017 \\ &= \frac{8y^3}{2015y^3 + 2y(y - \sqrt{y}) - (y^2 + y - 2y\sqrt{y})} \\ &= \frac{8y^3}{2015y^3 + y^2 - y} \\ &= \frac{8y^2}{2015y^2 + y - 1} \end{aligned}$$

so that

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{8(2017)^2}{2015(2017)^2+2016} = \frac{32546312}{8197604351}.$$

Solution 4 by Arkady Alt, San Jose, CA

Let $x = \dfrac{a+\sqrt{a-\sqrt{a}}}{a-\sqrt{a-\sqrt{a}}}$. Then, $x+\dfrac{1}{x} = \dfrac{a+\sqrt{a-\sqrt{a}}}{a-\sqrt{a-\sqrt{a}}} + \dfrac{a-\sqrt{a-\sqrt{a}}}{a+\sqrt{a-\sqrt{a}}} =$
$\dfrac{\left(a+\sqrt{a-\sqrt{a}}\right)^2+\left(a-\sqrt{a-\sqrt{a}}\right)^2}{a^2-a+\sqrt{a}} = \dfrac{2\left(a^2+a-\sqrt{a}\right)}{a^2-a+\sqrt{a}} = \dfrac{2\left(-a^2+a-\sqrt{a}+2a^2\right)}{a^2-a+\sqrt{a}} =$
$-2+\dfrac{4a^2}{a^2-a+\sqrt{a}} \iff x+\dfrac{1}{x}+2 = \dfrac{4a^2}{a^2-a+\sqrt{a}}$ and, therefore,
$\dfrac{(x+1)^4}{x((a-1)\,x^2-2x+(a-1))} = \dfrac{(x+1)^4}{x^2((a-1)\left(x+\dfrac{1}{x}+2\right)-2a)} =$

$\dfrac{\left(x+\dfrac{1}{x}+2\right)^2}{(a-1)\left(x+\dfrac{1}{x}+2\right)-2a} = \dfrac{\left(\dfrac{4a^2}{a^2-a+\sqrt{a}}\right)^2}{(a-1)\cdot\dfrac{4a^2}{a^2-a+\sqrt{a}}-2a} =$
$\dfrac{16a^4}{((a-1)\cdot 4a^2-2a\,(a^2-a+\sqrt{a}))\,(a^2-a+\sqrt{a})} = \dfrac{16a^4}{2a\,(a^2-a-\sqrt{a})\,(a^2-a+\sqrt{a})} =$
$\dfrac{8a^3}{(a^2-a)^2-a} = \dfrac{8a^2}{a\,(a-1)^2-1}.$

For $a=2017$ we get $\dfrac{(x+1)^4}{x(2016x^2-2x+2016)} = \dfrac{8\cdot 2017^2}{2017\cdot 2016^2-1}.$

Solution 5 by Kee-Wai Lau, Hong Kong, China

We show that

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{32546312}{8197604351} \qquad (1)$$

Firstly we have

$$\begin{aligned}
x+\frac{1}{x} &= \frac{2017+\sqrt{2017-\sqrt{2017}}}{2017-\sqrt{2017-\sqrt{2017}}} + \frac{2017-\sqrt{2017-\sqrt{2017}}}{2017+\sqrt{2017-\sqrt{2017}}} \\
&= \frac{\left(2017+\sqrt{2017-\sqrt{2017}}\right)^2+\left(2017-\sqrt{2017-\sqrt{2017}}\right)^2}{\left(2017-\sqrt{2017-\sqrt{2017}}\right)^2+\left(2017+\sqrt{2017-\sqrt{2017}}\right)^2} \\
&= \frac{2(4070306-\sqrt{2017})}{4066272+\sqrt{2017}} \\
&= \frac{2(4070306-\sqrt{2017})(4066272-\sqrt{2017})}{(4066272+\sqrt{2017})(4066272-\sqrt{2017})}
\end{aligned}$$

$$= \frac{2(8205736897 - 4034\sqrt{2017})}{8197604351}.$$

Next, we have

$$\left(x + \frac{1}{x} + 2\right)^2 = \frac{131291822608(8197604353 - 4032\sqrt{2017})}{67200717095534131201}$$

and

$$2016\left(x + \frac{1}{x}\right) - 2 = \frac{4034(8197604353 - 4032\sqrt{2017})}{8197604351}.$$

Since $\dfrac{(x+1)^4}{x(2016x^2 - 2x + 2016)} = \dfrac{\left(x + \frac{1}{x} + 2\right)^2}{2016\left(x + \frac{1}{x}\right) - 2}$, so (1) follows.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Telman Rashidov, Azerbaijan Medical University, Baku Azerbaijan; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5428: *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

If $x > 0$, then $\dfrac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \dfrac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} \geq 1 - \dfrac{1}{\sqrt[4]{2}}$, where [.] and {.} respectively denote the integer part and the fractional part of x.

Solution 1 by Soumava Chakraborty, Kolkata, India

Case 1: $0 < x < 1 \quad [x] = 0$. Therefore,

$$LHS = \frac{\{x\}}{\sqrt[4]{17\{x\}^4}} = \frac{1}{\sqrt[4]{17}} > 1 - \frac{1}{\sqrt[4]{2}}.$$

Case 2: $[x] \geq 1$ and $\{x\} = 0$. Therefore,

$$LHS = \frac{[x]}{\sqrt[4]{2[x]^4}} = \frac{1}{\sqrt[4]{2}} > 1 - \frac{1}{\sqrt[4]{2}}.$$

Case 3: $[x] \geq 1$ and $0 < \{x\} < 1$. Therefore,

$$\begin{aligned} \{x\} < 1 \leq [x] \quad &\Rightarrow \quad \{x\} < [x]\Big(2\{x\} + [x]\Big)^4 + [x]^4 < 82[x]^4 \\ &\Rightarrow \quad \frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} > \frac{1}{\sqrt[4]{82}}, \text{and } \frac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} > 0, \text{ and therefore} \\ LHS \quad &> \quad \frac{1}{\sqrt[4]{82}} > 1 - \frac{1}{\sqrt[4]{2}}. \end{aligned}$$

Combining the 3 cases, the LHS is always $> \dfrac{1}{\sqrt[4]{82}}$ which is $> 1 - \dfrac{1}{\sqrt[4]{2}}$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since $x = [x] + \{x\}$ and $[x] \leq x < [x] + 1$, we have that $[x] + 2\{x\} = x + \{x\}$ and $\{x\} = x - [x] < 1$, so $[x] + 2\{x\} = x + \{x\} \leq x + x = 2x$ and, thus, since $x > 0$, $(x + \{x\})^4 < (2x)^4$; hence, $[x]^4 + (x + \{x\})^4 < x^4 + 16x^4$ and $\{x\}^4 + (x + \{x\})^4 < x^4 + 16x^4$.

It follows that $0 < \sqrt[4]{[x]^4 + (x + \{x\})^4} < \sqrt[4]{17x^4}$ and $0 < \sqrt[4]{\{x\}^4 + (x + \{x\})^4} < \sqrt[4]{17x^4}$ so

$0 < \dfrac{1}{\sqrt[4]{[x]^4 + (x + \{x\})^4}} \leq \dfrac{1}{\sqrt[4]{17}x}$ and $0 < \dfrac{1}{\sqrt[4]{\{x\}^4 + (x + \{x\})^4}} \leq \dfrac{1}{\sqrt[4]{17}x}$ and hence,

$\dfrac{[x]}{\sqrt[4]{\{x]\}^4 + (x + \{x\})^4}} \leq \dfrac{[x]}{\sqrt[4]{17}x}$ with equality iff $[x] = 0$ and
$0 < \dfrac{\{x\}}{\sqrt[4]{\{x]\}^4 + (x + \{x\})^4}} \leq \dfrac{\{x\}}{\sqrt[4]{17}x}$ with equality iff $\{x\} = 0$, so

$$\frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \frac{\{x\}}{\sqrt[4]{[\{x\}^4 + ([x] + 2\{x\})^4}} = \frac{[x]}{\sqrt[4]{[x]^4 + (x + \{x\})^4}} + \frac{\{x\}}{\sqrt[4]{[\{x\}^4 + (x + \{x\})^4}}$$

$$\geq \quad \frac{[x]}{\sqrt[4]{17}x} + \frac{\{x\}}{\sqrt[4]{17}x} = \frac{[x] + \{x\}}{\sqrt[4]{17}x} = \frac{x}{\sqrt[4]{17}x} = \frac{1}{\sqrt[4]{17}}$$

with equality iff $[x] = 0$ and $\{x\} = 0$, that is, iff $0 < x < 1$ and $x \in N$, with is impossible.

Hence, we have proved the more general and strict inequality

$$\frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \frac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} > \frac{1}{\sqrt[4]{17}}$$

(which implies, because $\dfrac{1}{\sqrt[4]{17}} + \dfrac{1}{\sqrt[4]{2}} = 1.33338\cdots > 1$, the initial result.)

Also solved by Moti Levy, Rehovot, Israel; Nirapada Pal-India, and the proposer.

5429: *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Prove that there are infinitely many positive integers a, b such that $18a^2 - b^2 - 6a - b = 0$.

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Define

$$g(a, b) = 18a^2 - 6a - b^2 - b$$

and

$$f(a, b) = (577a + 136b - 28, 2448a + 577b - 120).$$

By direct computation we see that $g(f(a,b)) = g(a,b)$. If $g(a_0, b_0) = 0$ with $a_0, b_0 \in N$ then the iterates $(a_n, b_n) = f(a_{n-1}, b_{n-1})$ are in $N \times N$ and satisfy $g(a_n, b_n) = 0$, for all $n \in N$.

Since $g(1,3) = 0$, starting with $(a_0, b_0) = (1,3)$ we obtain the infinite sequence of solutions

$$(1,3), \quad (957, 4059), \quad (1104185, 4684659), \quad (1274228341, 5406093003),$$
$$(1470458401137, 6238626641379), \ldots$$

Since $g(5,20) = 0$, starting with $(a_0, b_0) = (5,20)$ we obtain another infinite sequence of solutions:

$$(5,20), \quad (5577, 23660), \quad (6435661, 27304196), \quad (7426747025, 31509019100),$$
$$(8570459630997, 36361380737780), \ldots$$

Solution 2 by Trey Smith, Angelo State University, San Angelo, TX

Solution by Trey Smith, Angelo State University, San Angelo, TX 76909

We start by observing that

$$18a^2 - b^2 - 6a - b = 0 \quad \Rightarrow \quad (2b+1)^2 - 2(6a-1)^2 = -1$$

which is suspiciously close to being Pell's Equation. Our particular equation is of the form

$$x^2 - 2y^2 = -1.$$

Notice that $(7,5)$ ($x = 7$ and $y = 5$) is a solution to $x^2 - 2y^2 = -1$. We will now create a sequence of solutions starting with $(c_0, d_0) = (7,5)$ in the following recursive manner. For $n \geq 0$, let

$$c_{n+1} = c_n^3 + 6c_n d_n^2, \quad d_{n+1} = 3c_n^2 d_n + 2d_n^3.$$

We prove the following facts regarding this sequence.

Fact 1. For all n, (c_n, d_n) is a solution to $x^2 - 2y^2 = -1$.

Proof: We use induction to prove this. In the ground case, it is clear that $(c_0, d_0) = (7,5)$ is a solution to $x^2 - 2y^2 = -1$.

Assume that (c_n, d_n) is a solution.

$$
\begin{aligned}
& c_{n+1}^2 - 2d_{n+1}^2 \\
= \; & (c_n^3 + 6c_n d_n^2)^2 - 2(3c_n^2 d_n + 2d_n^3)^2 \\
= \; & c_n^6 + 12c_n^4 d_n^2 + 36c_n^2 d_n^4 - 2(9c_n^4 d_n^2 + 12c_n^2 d_n^4 + 4d_n^6) \\
= \; & c_n^6 + 12c_n^4 d_n^2 + 36c_n^2 d_n^4 - 18c_n^4 d_n^2 - 24c_n^2 d_n^4 - 8d_n^6 \\
= \; & c_n^6 - 6c_n^4 d_n^2 + 12c_n^2 d_n^4 - 8d_n^6 \\
= \; & (c_n^2 - 2d_n^2)^3 \\
= \; & -1.
\end{aligned}
$$

For the next two facts, we use the notation $q \equiv_m t$ to represent the statement $q \equiv t \pmod{m}$.

Fact 2. For all n, $c_n \equiv_3 1$ and $c_n \equiv_2 1$.

Proof: We proceed by induction noting, first, that $c_0 \equiv_3 1$ and $c_0 \equiv_2 1$. Then assuming that $c_n \equiv_3 1$ we have that

$$c_{n+1} = c_n^3 + 6c_n d_n^2 \equiv_3 1^3 + 0 = 1.$$

Also, assuming that $c_n \equiv_2 1$, we have

$$c_{n+1} = c_n^3 + 6c_n d_n^2 \equiv_2 1^3 + 0 = 1.$$

Fact 3. For all n, $d_n \equiv_2 1$.

Proof: Clearly $d_0 \equiv_2 1$. Assuming that $d_n \equiv_2 1$, we have

$$d_{n+1} = 3c_n^2 d_n + 2d_n^3 \equiv_2 3 \cdot 1^2 \cdot 1 + 0 = 3 \equiv_2 1.$$

Fact 4. For all n, $d_{2n} \equiv_3 2$.

Proof: Certainly $d_0 \equiv_3 2$. Assume that for n, $d_{2n} \equiv_3 2$. Then

$$d_{2n+1} = 3c_{2n}^2 d_{2n} + 2d_{2n}^3 \equiv_3 0 + 2 \cdot 2^3 \equiv_3 1,$$

so that

$$d_{2(n+1)} = d_{2n+2} = 3c_{2n+1}^2 d_{2n+1} + 2d_{2n+1}^3 \equiv_3 0 + 2 \cdot 1^3 \equiv_3 2.$$

Using the facts above, we show that there are infinitely many pairs (a, b) that satisfy $(2b+1)^2 - 2(6a-1)^2 = -1$. Fix an even number m. Then (c_m, d_m) satisfies $x^2 - 2y^2 = -1$. Since $c_m \equiv_2 1$ we have that $c_m - 1$ is even (and greater than 0) so that

$$b = \frac{c_m - 1}{2}$$

is an integer. Also, $d_m \equiv_3 2$ which tells us that $d_m + 1$ is divisible by 3, and since $d_m \equiv_2 1$, $d_m + 1$ is divisible by 2. Hence $d_m + 1$ is divisible by 6. Then

$$a = \frac{d_m + 1}{6}$$

is an integer. Thus, the pair (a, b) is a solution to $18a^2 - b^2 - 6a - b = 0$.

Solution 3 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Observe two such solutions (a, b) are given by $(1, 3)$ and $(5, 20)$. We claim that if (a_i, b_i) is a solution in positive integers, then so is (a_{i+1}, b_{i+1}) where

$$\begin{aligned} a_{i+1} &= 577a_i + 136b_i - 28 \\ b_{i+1} &= 2448a_i + 577b_i - 120. \end{aligned}$$

To see this, note that (a_{i+1}, b_{i+1}) are clearly positive integers and

$$\begin{aligned} 18a_{i+1}^2 - b_{i+1}^2 - 6a_{i+1} - b_{i+1} &= 18(577a_i + 136b_i - 28)^2 - (2448a_i + 577b_i - 120)^2 \\ &\quad -6(577a_i + 136b_i - 28) - (2448a_i + 577b_i - 120) \\ &= 18a_i^2 - b_i^2 - 6a_i - b_i \\ &= 0. \end{aligned}$$

The solutions $(1, 3)$ and $(5, 20)$ are seeds which produce two infinite families of solutions. The first four solutions in each family is given below.

i	a_i	b_i	a_i	b_i
1	1	3	5	20
2	957	4059	5577	23660
3	1104185	4684659	6435661	27304196
4	1274228341	5406093003	7426747025	31509019100

Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The proposed equation may be written as follows:

$$\begin{aligned} 18a^2 - b^2 - 6a - b &= 0 \\ 18\left(a - \frac{1}{6}\right)^2 - \frac{1}{2} - \left(b + \frac{1}{2}\right)^2 + \frac{1}{4} &= 0 \\ 18\left(a - \frac{1}{6}\right)^2 - \left(b + \frac{1}{2}\right)^2 &= \frac{1}{4} \\ 72\left(a - \frac{1}{6}\right)^2 - 4\left(b + \frac{1}{2}\right)^2 &= 1 \\ (2b+1)^2 - 2(6a-1)^2 &= -1. \end{aligned}$$

The last equation is a Pell-type equation $x^2 - 2y^2 = -1$, by doing $x = 2b + 1$ and $y = 6a - 1$. The smallest solution of $x^2 - 2y^2 = -1$ is $(1, 1)$ and therefore all its solutions are given by $x_i + y_i\sqrt{2} = \left(1 + \sqrt{2}\right)^{2i+1}$. Note that x_i and y_i are allways odd so b is an integer. Also $6a = 1 + \sum_{k\geq 0} \binom{2i+1}{2k+1}$. Since the expression $1 + \sum_{k\geq 0} \binom{2i+1}{2k+1}$ is even and multiple of 3 for i of the form $i = 6m - 1$, for m integer, the proposed equation has infinitely many positive integral solutions.

Solution 5 by David E. Manes, SUNY at Oneonta, NY

Solution. Writing the equation as a quadratic in b, one obtains $b^2 + b - 6a(3a - 1) = 0$ and, since we want positive integer solutions,

$$b = \frac{-1 + \sqrt{1 + 72a^2 - 24a}}{2}.$$

Note that the above fraction is a positive integer provided that $72a^2 - 24a + 1 = c^2$ for some integer c. This last equation is equivalent to a negative Pell equation $c^2 - 2d^2 = -1$, where $d = 6a - 1$. This equation is solvable and the positive integer solutions are given by the odd powers of $1 + \sqrt{2}$. More precisely, if n is a positive integer and (c_n, d_n) is a solution of $c^2 - 2d^2 = -1$, then $c_n + d_n\sqrt{2} = (1 + \sqrt{2})^{2n-1}$. The problem is that not all the solutions for d_n yield solutions for a_n.

Observe: 1) if $n \equiv 0 \pmod 4$, then $c_n \equiv 5 \pmod 6$ and $d_n \equiv 1 \pmod 6$, 2) if $n \equiv 1 \pmod 4$, then $c_n \equiv d_n \equiv 1 \pmod 6$, 3) if $n \equiv 2 \pmod 4$, then $c_n \equiv 1 \pmod 6$ and $d_n \equiv 5 \pmod 6$, 4) if $n \equiv 3 \pmod 4$, then $c_n \equiv d_n \equiv 5 \pmod 6$.

The above observations provide straightforward inductive arguments for the following consequences. If $n \equiv 0$ or $1 \pmod 4$, then there are no solutions since $d_n \equiv 1 \pmod 6$ implies no integer solution for a_n. On the other hand, if $n \equiv 2$ or $3 \pmod 4$, then $a_n = \dfrac{d_n + 1}{6}$ is a positive integer and $b_n = (-1 + \sqrt{72a_n^2 - 24a_n + 1})/2$. Since there are infinitely many positive integers congruent to 2 or 3 modulo 4, the result follows.

Some of the infinitely many solutions are: if $n = 2$, then $c_2 = 7, d_2 = 5$ and $(a_2, b_2) = (1, 3)$; if $n = 3$, then $c_3 = 41, d_3 = 29$ and $(a_3, b_3) = (5, 20)$; if $n = 6$, then $c_6 = 8119, d_6 = 5741$ and $(a_6, b_6) = (957, 4059)$; if $n = 7$, then $c_7 = 47321, d_7 = 33461$ and $(a_7, b_7) = (5577, 23660)$; if $n = 10$, then $c_{10} = 9369319, d_{10} = 6625109$ and $(a_{10}, b_{10}) = (1104185, 4684659)$; if $n = 11$, then $c_{11} = 54608393, d_{11} = 38613965$ and $(a_{11}, b_{11}) = (6435661, 27304196)$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Anthony J. Bevelacqua, University of North Dakota, ND; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Kenneth Korbin, NY, NY; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5430: *Proposed by Oleh Faynshteyn, Leipzig, Germany*

Let a, b, c be the side-lengths, α, β, γ the angles, and R, r the radii respectively of the circumcircle and incircle of a triangle. Show that

$$\frac{a^3 \cdot \cos(\beta - \gamma) + b^3 \cdot \cos(\gamma - \alpha) + c^3 \cdot \cos(\alpha - \beta)}{(b + c)\cos\alpha + (c + a)\cos\beta + (a + b)\cos\gamma} = 6Rr.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

By the Law of Cosines,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

and hence,

$$(b + c)\cos\alpha = \frac{(b + c)\left(b^2 + c^2 - a^2\right)}{2bc} = \frac{a\,(b + c)\left(b^2 + c^2 - a^2\right)}{2abc}.$$

Similarly,

$$(c+a)\cos\beta = \frac{b\,(c+a)\left(c^2+a^2-b^2\right)}{2abc}$$

and

$$(a+b)\cos\gamma = \frac{c\,(a+b)\left(a^2+b^2-c^2\right)}{2abc}.$$

Therefore,

$$\begin{aligned}
&(b+c)\cos\alpha + (c+a)\cos\beta + (a+b)\cos\gamma \\
&= \frac{a\,(b+c)\left(b^2+c^2-a^2\right)+b\,(c+a)\left(c^2+a^2-b^2\right)+c\,(a+b)\left(a^2+b^2-c^2\right)}{2abc} \\
&= \frac{2a^2bc+2ab^2c+2abc^2}{2abc} \\
&= a+b+c. \qquad (1)
\end{aligned}$$

If K is the area of the given triangle, then

$$K = \frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta$$

and we have

$$\sin\alpha = \frac{2K}{bc}, \quad \sin\beta = \frac{2K}{ca}, \quad \text{and} \quad \sin\gamma = \frac{2K}{ab}.$$

Thus,

$$\begin{aligned}
a^3\cos(\beta-\gamma) &= a^3\left[\cos\beta\cos\gamma+\sin\beta\sin\gamma\right] \\
&= a^3\left[\frac{\left(c^2+a^2-b^2\right)}{2ca}\cdot\frac{\left(a^2+b^2-c^2\right)}{2ab}+\frac{4K^2}{(ca)\,(ab)}\right] \\
&= a\left[\frac{a^4-\left(b^2-c^2\right)^2+16K^2}{4bc}\right] \\
&= \frac{a^2}{4abc}\left[a^4-\left(b^2-c^2\right)^2+16K^2\right].
\end{aligned}$$

By Heron's Formula,

$$\begin{aligned}
16K^2 &= (a+b+c)\,(a+b-c)\,(b+c-a)\,(c+a-b) \\
&= \left[(a+b)^2-c^2\right]\left[c^2-(a-b)^2\right] \\
&= 2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
a^3\cos(\beta-\gamma) &= \frac{a^2}{4abc}\left[a^4-\left(b^2-c^2\right)^2+2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)\right] \\
&= \frac{a^2}{4abc}\left[-2b^4-2c^4+2\left(a^2b^2+2b^2c^2+c^2a^2\right)\right] \\
&= \frac{a^2}{2abc}\left(-b^4-c^4+a^2b^2+2b^2c^2+c^2a^2\right).
\end{aligned}$$

Similarly,

$$b^3\cos(\gamma-\alpha) = \frac{b^2}{2abc}\left(-c^4-a^4+a^2b^2+b^2c^2+2c^2a^2\right)$$

and

$$c^3\cos(\alpha-\beta)=\frac{c^2}{2abc}\left(-a^4-b^4+2a^2b^2+b^2c^2+c^2a^2\right).$$

As a result,

$$\begin{aligned}
&a^3\cos(\beta-\gamma)+b^3\cos(\gamma-\alpha)+c^3\cos(\alpha-\beta)\\
&=\frac{a^2}{2abc}\left(-b^4-c^4+a^2b^2+2b^2c^2+c^2a^2\right)+\frac{b^2}{2abc}\left(-c^4-a^4+a^2b^2+b^2c^2+2c^2a^2\right)\\
&+\frac{c^2}{2abc}\left(-a^4-b^4+2a^2b^2+b^2c^2+c^2a^2\right)\\
&=\frac{1}{2abc}\cdot 6a^2b^2c^2\\
&=3abc. \qquad (2)
\end{aligned}$$

By (1) and (2),

$$\frac{a^3\cos(\beta-\gamma)+b^3\cos(\gamma-\alpha)+c^3\cos(\alpha-\beta)}{(b+c)\cos\alpha+(c+a)\cos\beta+(a+b)\cos\gamma}=\frac{3abc}{a+b+c}. \qquad (3)$$

Finally, if $s=\dfrac{a+b+c}{2}$, then

$$R=\frac{abc}{4K} \quad \text{and} \quad K=rs$$

and we get

$$\begin{aligned}
6Rr&=6\left(\frac{abc}{4K}\right)\left(\frac{K}{s}\right)\\
&=\frac{3abc}{2s}\\
&=\frac{3abc}{a+b+c}. \qquad (4)
\end{aligned}$$

Conditions (3) and (4) yield the desired result.

Solution 2 by Moti Levy, Rehovot, Israel

After substituting $Rr=\frac{abc}{2(a+b+c)}$ in the right hand side of the original inequality, it becomes

$$\frac{\sum_{cyc}a^3\cos(\beta-\gamma)}{\sum_{cyc}(b+c)\cos\alpha}=\frac{3abc}{a+b+c}.$$

Thus, we actually need to prove two identities (which appeared many times before in the literature):

$$\sum_{cyc}(b+c)\cos\alpha=a+b+c, \qquad (1)$$

$$\sum_{cyc}a^3\cos(\beta-\gamma)=3abc. \qquad (2)$$

Dropping a perpendicular from C to side c, it divides the triangle into two right triangles, and c into two pieces $c = a\cos\beta + b\cos\alpha$, and similarly for all sides:

$$\begin{aligned} c &= a\cos\beta + b\cos\alpha, \\ a &= b\cos\gamma + c\cos\beta, \\ b &= c\cos\alpha + a\cos\gamma. \end{aligned}$$

To prove (1), we add the three equations, and get immediately:

$$a+b+c = a\cos\beta + b\cos\alpha + b\cos\gamma + c\cos\beta + c\cos\alpha + a\cos\gamma = \sum_{cyc}(b+c)\cos\alpha.$$

To prove (2), we use the following trigonometric identity

$$\cos(x-y) = \frac{\sin x\cos x + \sin y\cos y}{\sin(x+y)},$$

and the triangle identity

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

$$\begin{aligned} a^3\cos(\beta-\gamma) &= a^3\frac{\sin\beta\cos\beta + \sin\gamma\cos\gamma}{\sin(\beta+\gamma)} \\ &= a^3\frac{\sin\beta\cos\beta + \sin\gamma\cos\gamma}{\sin\alpha} \\ &= a^3\frac{b\cos\beta + c\cos\gamma}{a} = a^2b\cos\beta + a^2c\cos\gamma \end{aligned}$$

$$\begin{aligned} \sum_{cyc} a^3\cos(\beta-\gamma) &= \sum_{cyc}\left(a^2b\cos\beta + a^2c\cos\gamma\right) \\ &= ab(a\cos\beta + b\cos\alpha) + ac(c\cos\alpha + a\cos\gamma) + bc(b\cos\gamma + c\cos\beta) \\ &= 3abc. \end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Kevin Soto Palacios, Huarmey, Peru; Neculai Stanciu, "Geroge Emil Palade" School Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5431: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let F_n be the n^{th} Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all $n \geq 3$, $\mathrm{F}_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{n=1}^{\infty}\left(\frac{1}{11}\right)^{F_nF_{n+1}}$$

is an irrational number and determine it (*).

The asterisk $(*)$ indicates that neither the author of the problem nor the editor are aware of a closed form for the irrational number.

Solution 1 by Moti Levy, Rehovot, Israel

It is well known that

$$F_nF_{n+1}=\sum_{k=1}^{n}F_k^2, \qquad (1)$$

hence $x:=\sum_{n=1}^{\infty}\left(\frac{1}{11}\right)^{F_nF_{n+1}}$ can be expressed as

$$x=\frac{1}{11^{F_1^2}}+\frac{1}{\left(11^{F_1^2}\right)\left(11^{F_2^2}\right)}+\frac{1}{\left(11^{F_1^2}\right)\left(11^{F_2^2}\right)\left(11^{F_3^2}\right)}+\cdots,$$

or

$$x=\sum_{k=1}^{\infty}\frac{1}{a_1a_2\cdots a_k}, \qquad a_k=11^{F_k^2}. \qquad (2)$$

The series (2) is the *Engel expansion* of the positive real number x. See [1] for definition of Engel expansion.

In 1913, Engel established the following result (See [2] page 303):

Every real number x has a unique representation $c+\sum_{k=1}^{\infty}\frac{1}{a_1a_2\cdots a_k}$, where c is an integer and $2\le a_1\le a_2\le a_3\le\cdots$ is a sequence of integers. Conversely, every such sequence is convergent and its sum is irrational if and only if $\lim_{k\to\infty}a_k=\infty$.
Therefore, by Engel's result, $\sum_{n=1}^{\infty}\frac{1}{11^{F_nF_{n+1}}}$ is irrational, since $\lim_{k\to\infty}11^{F_k^2}=\infty$.

I do not know how to express x in closed form. However, it can be shown that it is *transcendental.* To this end, I rely on a result given in [2] (on page 315):

Let $(f(n))_{n\ge1}$ be a sequence of positive integers such that $\lim_{n\to\infty}\frac{f(n+1)}{f(n)}=\mu>2$. Then for every integer $d\ge2$, the number $x=\sum_{n=1}^{\infty}\frac{1}{d^{f(n)}}$ is transcendental.

In our case, $d=11$ and $f(n)=F_nF_{n+1}$. We check that

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n+1)}{f(n)}&=\lim_{n\to\infty}\frac{F_{n+1}F_{n+2}}{F_nF_{n+1}}=\lim_{n\to\infty}\frac{F_{n+2}}{F_n}=\lim_{n\to\infty}\frac{F_{n+1}+F_n}{F_n}\\&=1+\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{3+\sqrt{5}}{2}\cong2.618>2.\end{aligned}$$

Then $x=\sum_{n=1}^{\infty}\frac{1}{11^{F_nF_{n+1}}}$ is transcendental.

References:

[1] Wikipedia "Engel expansion".

[2] Ribenboim Paulo, "*My Numbers, My Friends: Popular Lectures on Number Theory*", Springer 2000.

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

Let p be a prime. For the sake of brevity put $c_k=F_kF_{k+1}$. We prove that the number

$$s=\sum_{k=1}^{\infty}\left(\frac{1}{p}\right)^{c_k}$$

is transcendental, in particular irrational.

The partial sum
$$s_n = \sum_{k=1}^{n} \left(\frac{1}{p}\right)^{c_k} = \frac{a_n}{b_n}$$
with positive integers a_n and $b_n \le p^{c_n}$ satisfies
$$\begin{aligned} 0 \quad < \quad s - s_n &= \sum_{k=n+1}^{\infty} \left(\frac{1}{p}\right)^{c_k} \le \left(\frac{1}{p}\right)^{c_{n+1}} \sum_{k=0}^{\infty} \left(\frac{1}{p}\right)^{k} \\ &= \frac{1}{p-1}\left(\frac{1}{p}\right)^{c_{n+1}-1} \le \frac{1}{(p^{c_n})^{\frac{c_{n+1}-1}{c_n}}}, \end{aligned}$$
because $c_{k+1} - c_k = F_{k+1}F_{k+2} - F_kF_{k+1} = F_{k+1}^2 \ge 1$. Since
$$\lim_{n\to\infty} \frac{c_{n+1}-1}{c_n} = \lim_{n\to\infty} \frac{F_{n+1}F_{n+2}-1}{F_nF_{n+1}} = \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n} \cdot \frac{F_{n+2}}{F_{n+1}}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} > 2$$
By the theorem of Thue, Siegel and Roth, for any (fixed) algebraic number x and $\varepsilon > 0$, the inequality
$$0 < \left| x - \frac{a}{b} \right| < \frac{1}{b^{2+\varepsilon}}$$
is satisfied only by a finite number of integers a and b. Hence, s is transcendental.

Also solved by the Kee-Wai Lau, Hong Kong, China (first part of the problem), and the proposer, (first part of the problem)

5432: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$, with $f(1) = \sqrt{2}$, such that
$$f'\left(\frac{1}{x}\right) = \frac{1}{f(x)}, \ \forall x > 0.$$

Solution 1 by Arkady Alt, San Jose, CA

First note that $f'\left(\frac{1}{x}\right) = \frac{1}{f(x)}, \ \forall x > 0 \iff f'(x) = \frac{1}{f\left(\frac{1}{x}\right)}, \ \forall x > 0.$

Then, since $f''(x) = \left(\frac{1}{f\left(\frac{1}{x}\right)}\right)' = -\frac{f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{f^2\left(\frac{1}{x}\right)}$ and

$\frac{1}{f^2\left(\frac{1}{x}\right)} = (f'(x))^2, f'\left(\frac{1}{x}\right) = \frac{1}{f(x)},$

we obtain $f''(x) = \frac{1}{x^2}(f'(x))^2 \frac{1}{f(x)} \iff \frac{f(x)f''(x)}{(f'(x))^2} = \frac{1}{x^2} \iff$

$\frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} - 1 = -\frac{1}{x^2} \iff$

$$\left(\frac{f(x)}{f'(x)}\right)' = 1-\frac{1}{x^2} \iff \frac{f(x)}{f'(x)} = x+\frac{1}{x}+c \iff \frac{f'(x)}{f(x)} = \frac{x}{x^2+cx+1}.$$

Since $f'(1) = \dfrac{1}{f(1)} = \dfrac{1}{\sqrt{2}}$ then $\dfrac{f(1)}{f'(1)} = 1+\dfrac{1}{1}+c \iff 2 = 2+c \iff c=0.$

Therefore, $\dfrac{f(x)}{f'(x)} = x+\dfrac{1}{x} \iff \dfrac{f'(x)}{f(x)} = \dfrac{x}{x^2+1} \iff \ln f(x) = \dfrac{1}{2}\ln\left(x^2+1\right)+d$ and, using $f(1)=\sqrt{2}$
again, we obtain $\ln f(1) = \dfrac{1}{2}\ln\left(1^2+1\right)+d \iff \ln\sqrt{2} = \dfrac{1}{2}\ln 2+d \iff d=0.$

Thus, $f(x) = \sqrt{x^2+1}$.

Solution 2 by Albert Stadler, Hirrliberg, Switzerland

The differential equation $f'(x) = \dfrac{1}{f\left(\frac{1}{x}\right)}$ shows that f is differentiable infinitely often in $x>0$. We differentiate the equation $f'(x)f\left(\dfrac{1}{x}\right) = 1$ and get

$$f''(x)f\left(\frac{1}{x}\right) - f'(x)f'\left(\frac{1}{x}\right)\frac{1}{x^2} = \frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)}\frac{1}{x^2} = 0,$$

or equivalently

$$\frac{f''(x)f(x)}{(f'(x))^2} = \frac{1}{x^2}. \qquad (1)$$

By assumption $f(1)=\sqrt{2}$ and thus $f'(1) = \dfrac{1}{f(1)} = \dfrac{\sqrt{2}}{2}$.

We integrate (1) and apply partial integration to get

$$\begin{aligned}
1-\frac{1}{x} = \int_1^x \frac{dt}{t^2} &= \int_1^x \frac{f''(t)f(t)}{(f'(t))^2}dt \\
&= \int_1^x \frac{d}{dt}\left(\frac{-1}{f'(t)}\right)f(t)dt \\
&= -\frac{f(t)}{f'(t)}\bigg|_1^x + \int_1^x \frac{f'(t)}{f'(t)}dt \\
&= \frac{f(1)}{f'(1)} - \frac{f(x)}{f'(x)} + x - 1 \\
&= 1 - \frac{f(x)}{f'(x)} + x.
\end{aligned}$$

So $\dfrac{f(x)}{f'(x)} = \dfrac{1}{x}+x$ or equivalently $\dfrac{f'(x)}{f(x)} = \dfrac{x}{1+x^2}$.

We integrate again and get

$$\ln f(x) - \ln f(1) = \int_1^x \frac{f'(t)}{f(t)}dt = \int_1^x \frac{t}{1+t^2}dt = \frac{1}{2}\ln(1+x^2) - = \frac{1}{2}\ln 2.$$

Therefore $f(x) = \sqrt{1+x^2}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $f : (0, +\infty) \to (0, +\infty)$ be a differentiable function that satisfies the hypothesis of the problem and let $g : (0, +\infty) \to (0, +\infty)$ be the differentiable function defined by $g(x) = \dfrac{1}{x}$. Since f is differentiable, and by the hypothesis $f'(x) = \dfrac{1}{(f \circ g)(x)}, \forall x > 0$, we conclude that f' is also differentiable and, differentiating both side of the equality $f'(x)f\left(\dfrac{1}{x}\right) = 1$, we obtain that $f''(x)f\left(\dfrac{1}{x}\right) + f'(x)f'\left(\dfrac{1}{x}\right)\dfrac{-1}{x^2} = 0$, and since $f\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2}$, or equivalently, $\dfrac{(f'(x))^2 - f''(x)f(x)}{(f'(x))^2} = 1 - \dfrac{1}{x^2}$, or what is the same, $\left(\dfrac{f}{f'}\right)'(x) = 1 - \dfrac{1}{x^2},\ \forall x > 0$.

Integrating both sides, we conclude that $\dfrac{f(x)}{f'(x)} = x + \dfrac{1}{x} + C,\ \forall x > 0$, for some $C \in \Re$. If we take $x = 1$ at the start of the inequality, and since $f(1) = \sqrt{2}$, we obtain that $f'(1) = \dfrac{1}{\sqrt{2}}$ and $\dfrac{f(1)}{f'(1)} = 2 + C$, from where $C = 0$, which implies, because $f(x) > 0\ \forall x > 0$ by hypothesis and $\dfrac{f(x)}{f'(x)} = x + \dfrac{1}{x} + 0$ and $\dfrac{f'(x)}{f(x)} = \dfrac{x}{x^2+1},\ \forall x > 0$. Integrating both sides of this last equality, we conclude that $\ln(f(x)) = \log\left(\sqrt{x^2+1}\right) + D,\ \forall x > 0$ for some $D \in \Re$. Taking $x = 1$ in this equality and using the fact that $f(1) = \sqrt{2}$, we find that $D = 0$ and therefore $f(x) = \sqrt{x^2+1},\ \forall x > 0$.

Since the function $f : (0, +\infty) \to (0, +\infty)$ defined by $f(x) = \sqrt{x^2+1}, \forall x > 0$, is differentiable with $f'(x) = \dfrac{x}{\sqrt{x^2+1}}$ and satisfies that $f(1) = \sqrt{2}$, and that $f\left(\dfrac{1}{x}\right) = \dfrac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}+1}} = \dfrac{1}{f(x)},\ \forall x > 0$, we conclude that the only differentiable function that satisfies the conditions of the problem is the function $f(x) = \sqrt{x^2+1},\ \forall x > 0$.

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

We have $f'\left(\frac{1}{x}\right)f(x) = 1$. Letting x to $\frac{1}{x}$ we also have $f'(x)f\left(\frac{1}{x}\right) = 1\ \ (*)$. Thus,

$$\begin{aligned}\frac{d}{dx}\left(f(x)f\left(\frac{1}{x}\right)\right) &= f'(x)f\left(\frac{1}{x}\right) + \left(-x^{-2}\right)f(x)f'\left(\frac{1}{x}\right)\\ &= 1 - x^{-2}.\end{aligned}$$

Integrating it, we have

$$f(x)\,f\left(\frac{1}{x}\right)=x+\frac{1}{x}+C$$

Letting $x=1$, we have $2=2+C$ or $C=0$. Therefore $f(x)\,f\left(\frac{1}{x}\right)=x+\frac{1}{x}$. Multiplying $f(x)$ to $(*)$, we have

$$\left(x+\frac{1}{x}\right)f'(x)=f(x)$$
$$\frac{f'(x)}{f(x)}=\frac{1}{x+\frac{1}{x}}$$

Integrating again, we have

$$\begin{aligned}\log f(x)&=\int\frac{dx}{x+\frac{1}{x}}\\&=\int\frac{x}{x^2+1}dx\\&=\frac{1}{2}\int\frac{\left(x^2+1\right)'}{x^2+1}dx\\&=\frac{1}{2}\log\left(x^2+1\right)+D\end{aligned}$$

Thus, we can write $f(x)=D\sqrt{x^2+1}$ where D is some constant. Letting $x=1$, we have $D=1$. Therefore, we have $f(x)=\sqrt{x^2+1}$, this function actually satisfies the condition.

Also solved by Abdallah El Farsi, Bechar, Algeria; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel; Moti Levy, Rehovot, Israel; Ravi Prakash, New Delhi, India, and the proposers.