Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* April 1, 2025.

• **5793** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Suppose $f : [a, b] \to [1, \infty)$ is a continuous function with $0 < a \leqslant b$. Then:

$$n(b-a)^{n-1}\int_a^b f(x)dx \leqslant (n-1)(b-a)^n + \left(\int_a^b f(x)dx\right)^n.$$

• **5794** *Proposed by Michel Bataille, Rouen, France.*

Let B_m denote the m-th Bernoulli number ($B_0 = 1$ and $(m+1)B_m + \sum_{j=0}^{m-1}\binom{m+1}{j}B_j = 0$ for $m \geqslant 1$). Evaluate

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}}.$$

• **5795** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate the integral

$$I = \int_0^\infty \frac{\arctan x}{(1+x)\sqrt{1+x^2}}dx.$$

• **5796** *Proposed by Problem proposed by Shivam Sharma, Delhi University, New Delhi, India, and Surjeet Singh, Indian Institute of Technology Kanpur, India.*

Here, ζ denotes the zeta function. Prove that:

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}}=\zeta(2).$$

• **5797** *Proposed by Toyesh Prakash Sharma and Etisha Sharma, Agra College, Agra, India.*

Solve the following differential equation without the aid of computers:

$$\left(x^2\ln^2 x\right)\frac{d^2y}{dx^2}-(2x\ln x)\frac{dy}{dx}+(2+\ln x)\,y+\ln^3 x=0.$$

• **5798** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Let a,b,c,d be positive real numbers such that $ab+bc+cd+da=4$. Prove that if (i) $a\geqslant b\geqslant 1\geqslant c\geqslant d$ or (ii) $a\geqslant b\geqslant c\geqslant 1\geqslant d$, then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+8\geqslant 3(a+b+c+d).$$

Solutions

To Formerly Published Problems

• **5769** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

Evaluate the limit $L=\lim\limits_{n\to\infty} n\,x_n$ where

$$x_n:=\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}}-n.$$

Solution 1 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

We will show that $L=-\dfrac{1}{6}$. First recall that for all $x\in\mathbb{R}$,

$$\sin(x)=\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

This alternating series thus implies that for $0\leqslant x\leqslant 1$,

$$0\leqslant x-\frac{1}{6}x^3\leqslant\sin(x)\leqslant x-\frac{1}{6}x^3+\frac{1}{120}x^5\leqslant x.$$

Hence, for all $n \geqslant 1$,

$$x_n \geqslant \frac{\frac{1}{n} - \frac{1}{6n^3}}{\frac{1}{n^2}} - n = -\frac{1}{6n}$$

and

$$x_n \leqslant \frac{\frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5}}{\frac{1}{n^2} - \frac{1}{6n^6}} - n = \frac{-n^3 + \frac{21}{20}n}{6n^4 - 1}.$$

Therefore

$$-\frac{1}{6} \leqslant nx_n \leqslant \frac{-n^4 + \frac{21}{20}n^2}{6n^4 - 1}$$

for all $n \geqslant 1$. It now follows from the sandwich theorem that $L = -\dfrac{1}{6}$.

Solution 2 by Devis Alvarado, Tegucigalpa, Honduras.

$$\begin{aligned}
L &= \lim_{n\to\infty} (nx_n) \\
&= \lim_{n\to\infty} \left(n \left(\frac{\sin\left(\frac{1}{n}\right)}{\sin\left(\frac{1}{n^2}\right)} - n \right) \right) \\
&= \lim_{x\to 0^+} \left(\frac{1}{x} \left(\frac{\sin(x)}{\sin(x^2)} - \frac{1}{x} \right) \right), \ x = \frac{1}{n}. \\
&= \lim_{x\to 0^+} \left(\frac{x\sin(x) - \sin\left(x^2\right)}{x^2 \sin(x^2)} \right) \\
&= \lim_{y\to 0^+} \left(\frac{2y\sin(2y) - \sin\left(4y^2\right)}{4y^2 \sin(4y^2)} \right), \ x = 2y \\
&= \lim_{y\to 0^+} \left(\frac{4y\sin(y)\cos(y) - 4\sin\left(y^2\right)\cos\left(y^2\right)\cos\left(2y^2\right)}{16y^2 \sin(y^2)\cos(y^2)\cos(2y^2)} \right) \\
&= \frac{1}{4} \lim_{y\to 0^+} \left(\frac{y\sin(y)\cos(y) - \sin\left(y^2\right)\cos(y) + \sin\left(y^2\right)\cos(y) - \sin\left(y^2\right)\cos\left(y^2\right)\cos\left(2y^2\right)}{y^2 \sin(y^2)\cos(y^2)\cos(2y^2)} \right) \\
&= \frac{1}{4} \lim_{y\to 0^+} \left(\frac{y\sin(y) - \sin\left(y^2\right)}{y^2\sin(y^2)} \cdot \frac{\cos(y)}{\cos(y^2)\cos(2y^2)} \right) \\
&+ \frac{1}{4} \lim_{y\to 0^+} \left(\frac{\cos(y) - \cos\left(y^2\right)\cos\left(2y^2\right)}{y^2} \cdot \frac{1}{\cos(y^2)\cos(2y^2)} \right) \\
&= \frac{1}{4}L - \frac{1}{8}
\end{aligned}$$

$$\Rightarrow L = -\frac{1}{6}.$$

$$
\begin{aligned}
\lim_{y\to 0^+}\left(\frac{\cos(y)-\cos\left(y^2\right)\cos\left(2y^2\right)}{y^2}\right) &= \lim_{y\to 0^+}\left(\frac{-\sin(y)-2y\sin\left(y^2\right)\cos\left(2y^2\right)-4y\cos\left(y^2\right)\sin\left(2y^2\right)}{2y}\right)\\
&= \lim_{y\to 0^+}\left(-\frac{\sin(y)}{2y}-\sin\left(y^2\right)\cos\left(2y^2\right)+2\cos\left(y^2\right)\sin\left(2y^2\right)\right)\\
&= -\frac{1}{2}.
\end{aligned}
$$

Solution 3 by Ilkin Hasanov ADA University. Baku, Azerbaijan.

Set $\frac{1}{n}=a$. Then $n\to\infty \iff a\to 0$. So

$$L=\lim_{n\to\infty} n\,x_n=\lim_{n\to\infty}\left(n\cdot\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}}-n^2\right)=\lim_{a\to 0}\left(\frac{1}{a}\cdot\frac{\sin a}{\sin a^2}-\frac{1}{a^2}\right)$$

Expressing the Taylor series for $\sin a$ and $\sin a^2$, we have

$$\sin a = a-\frac{a^3}{3!}+\frac{a^5}{5!}-\frac{a^7}{7!}+\dots$$

and

$$\sin a^2 = a^2-\frac{a^6}{3!}+\frac{a^{10}}{5!}-\frac{a^{14}}{7!}+\dots$$

Using the latter, we write

$$
\begin{aligned}
L=\lim_{a\to 0}\left(\frac{1}{a}\cdot\frac{\sin a}{\sin a^2}-\frac{1}{a^2}\right) &= \lim_{a\to 0}\frac{\left(a-\frac{a^3}{3!}+\frac{a^5}{5!}-\frac{a^7}{7!}+\dots\right)-\left(a-\frac{a^5}{3!}+\frac{a^9}{5!}-\frac{a^{13}}{7!}+\dots\right)}{a^3\left(1-\frac{a^4}{3!}+\frac{a^8}{5!}-\frac{a^{12}}{7!}+\dots\right)}\\
&= \lim_{a\to 0}\frac{\left(-\frac{1}{3!}+\frac{a^2}{5!}-\frac{a^4}{7!}+\dots\right)+\left(\frac{a^2}{3!}-\frac{a^6}{5!}+\frac{a^{10}}{7!}-\dots\right)}{1-\frac{a^4}{3!}+\frac{a^8}{5!}-\frac{a^{12}}{7!}+\dots}=\frac{-\frac{1}{6}}{1}=-\frac{1}{6}.
\end{aligned}
$$

Solution 4 by David A. Huckaby, Angelo State University, San Angelo, TX.

We rewrite

$$x_n=\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}}-n=\frac{\sin\frac{1}{n}-n\sin\frac{1}{n^2}}{\sin\frac{1}{n^2}}.$$

Now $\sin\frac{1}{n}=\frac{1}{n}-\frac{1}{6n^3}+\frac{1}{120n^5}+O(\frac{1}{n^7})$, $\sin\frac{1}{n^2}=\frac{1}{n^2}-\frac{1}{6n^6}+O(\frac{1}{n^{10}})$, and $n\sin\frac{1}{n^2}=\frac{1}{n}-\frac{1}{6n^5}+O(\frac{1}{n^9})$. So $\sin\frac{1}{n}-n\sin\frac{1}{n^2}=-\frac{1}{6n^3}+\frac{7}{40n^5}+O(\frac{1}{n^7})$. Thus

$$L = \lim_{n\to\infty} nx_n = \lim_{n\to\infty} n \frac{-\frac{1}{6n^3} + \frac{7}{40n^5} + O(\frac{1}{n^7})}{\frac{1}{n^2} - \frac{1}{6n^6} + O(\frac{1}{n^{10}})} = \lim_{n\to\infty} \frac{-\frac{1}{6n^2} + \frac{7}{40n^4} + O(\frac{1}{n^6})}{\frac{1}{n^2} - \frac{1}{6n^6} + O(\frac{1}{n^{10}})} = -\frac{1}{6}.$$

Solution 5 by Anthony Batiste (student) and the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

$L = -\dfrac{1}{6}$.

Let $u = \dfrac{1}{n}$, so that $u \to 0^+$ as $n \to \infty$. Using l'Hospital's Rule four times, we see that

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{n \sin \frac{1}{n}}{\sin \frac{1}{n^2}} - n^2 \\
&= \lim_{u\to 0^+} \frac{u \sin u - \sin u^2}{u^2 \sin u^2} \\
&= \lim_{u\to 0^+} \frac{u \cos u + \sin u - 2u \cos u^2}{2u^3 \cos u^2 + 2u \sin u^2} \\
&= \lim_{u\to 0^+} \frac{2\cos u - u \sin u + 4u^2 \sin u^2 - 2\cos u^2}{-4u^4 \sin u^2 + 10u^2 \cos u^2 + 2 \sin u^2} \\
&= \lim_{u\to 0^+} \frac{-3\sin u - u\cos u + 12u \sin u^2 + 8u^3 \cos u^2}{-8u^5 \cos u^2 - 36u^3 \sin u^2 + 24u \cos u^2} \\
&= \lim_{u\to 0^+} \frac{-4\cos u + u \sin u + 12 \sin u^2 + 48u^2 \cos u^2 - 16u^4 \sin u^2}{16u^6 \sin u^2 - 112u^4 \cos u^2 - 156u^2 \sin u^2 + 24 \cos u^2} \\
&= \frac{-4}{24} \\
&= -\frac{1}{6}.
\end{aligned}$$

Solution 6 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We have

$$\begin{aligned}
\frac{\sin \frac{1}{n}}{\sin \frac{1}{n^2}} &= \frac{\frac{1}{n} - \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right)}{\frac{1}{n^2} - \frac{1}{6n^5} + O\left(\frac{1}{n^10}\right)} = n - \frac{1}{6n} + O\left(\frac{1}{n^3}\right), \\
n\left(\frac{\sin \frac{1}{n}}{\sin \frac{1}{n^2}} - n\right) &= -\frac{1}{6} + O\left(\frac{1}{n^2}\right),
\end{aligned}$$

which has the limit $-1/6$ as $n \to \infty$.

Solution 7 by Michel Bataille, Rouen, France.

We have

$$nx_n = \frac{n}{\sin\frac{1}{n^2}}\left(\sin\frac{1}{n} - n\sin\frac{1}{n^2}\right)$$

and $\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + o(1/n^3)$ as $n \to \infty$. It first follows that

$$\frac{n}{\sin\frac{1}{n^2}} \sim \frac{n}{\frac{1}{n^2}} = n^3$$

and second that

$$\sin\frac{1}{n} - n\sin\frac{1}{n^2} = \frac{1}{n} - \frac{1}{6n^3} + o(1/n^3) - n\left(\frac{1}{n^2} + o(1/n^4)\right) = -\frac{1}{6n^3} + o(1/n^3)$$

so that $\sin\frac{1}{n} - n\sin\frac{1}{n^2} \sim -\frac{1}{6n^3}$.
As a result, we obtain

$$nx_n \sim n^3 \cdot \left(-\frac{1}{6n^3}\right) = -\frac{1}{6}$$

and so $L = -\frac{1}{6}$.

Solution 8 by Perfetti Paolo, Università di "Tor Vergata", Roma, Italy.

We know that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ hence

$$\lim_{n\to\infty}\frac{n\sin\frac{1}{n} - n^2\sin\frac{1}{n^2}}{\sin\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^3\sin\frac{1}{n} - n^4\sin\frac{1}{n^2}}{n^2\sin\frac{1}{n^2}} = \lim_{n\to\infty} n^3\sin\frac{1}{n} - n^4\sin\frac{1}{n^2}$$

Using $\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + o(\frac{1}{n^4})$

$$n^3\sin\frac{1}{n} - n^4\sin\frac{1}{n^2} = n^3\left(\frac{1}{n} - \frac{1}{6n^3} + o(\frac{1}{n^4})\right) - n^4\left(\frac{1}{n^2} - \frac{1}{6n^6} + o(\frac{1}{n^8})\right) =$$
$$= \frac{-1}{6} + o(\frac{1}{n}) \to \frac{-1}{6}$$

Solution 9 by Péter Fülöp, Gyömrő, Hungary.

Let's start from the Taylor series of $\sin(x)$:

$$\sin(\frac{1}{n}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n}\right)^{2k+1}$$

$$\sin(\frac{1}{n^2}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n^2}\right)^{2k+1}$$

So the limit is as follows

$$L = \lim_{n\to\infty} \frac{n\left(\frac{1}{n} - \frac{1}{6n^3} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n}\right)^{2k+1}\right)}{\frac{1}{n^2} - \frac{1}{6n^6} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n^2}\right)^{2k+1}} - n^2$$

After removing out the $\frac{1}{n^2}$ terms from all parts of the denominator and $\frac{1}{n^3}$ from the sum of the numerator, we get:

$$L = \lim_{n\to\infty} \frac{n\left(\frac{1}{n} - \frac{1}{6n^3} + \frac{1}{n^3}\sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n}\right)^{2k-2}\right)}{\frac{1}{n^2}\left(1 - \frac{1}{6n^4} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n^2}\right)^{2k-1}\right)} - n^2$$

Performing further transmissions:

$$L = \lim_{n\to\infty} \frac{n^2 - \frac{1}{6} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n}\right)^{2k-2} - n^2 + \frac{1}{6n^2} - \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n^2}\right)^{2k}}{1 - \frac{1}{6n^4} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{n^2}\right)^{2k-1}}$$

If $n \to \infty$ then all sums are go to zero, the n^2 terms are cancelled.

The value of the limit is

$$L = \lim_{n\to\infty} nx_n = -\frac{1}{6}.$$

Solution 10 by Yunyong Zhang, Chinaunicom, Yunnan, China.

We have the Taylor expansions:

$$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} + \cdots,$$
$$\sin\frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{6n^6} + \frac{1}{120n^10} + \cdots.$$

$$\therefore \quad \frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} = n - \frac{1}{6n} + \frac{7}{40n^3} + \cdots,$$

$$\therefore \quad L = \lim_{n\to\infty} n\left(-\frac{1}{6n} + \frac{7}{40n^3} + \cdots\right) = -\frac{1}{6}.$$

Also solved by Hong Biao Zeng, Fort Hays State University, Hays, KS and the problem proposer.

• **5770** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Suppose $a, b \in \mathbb{C}$ with $|a^2+1| \leqslant 1,\ |a^4+1| \leqslant 1,\ |b^3+1| \leqslant 1,\ |b^6+1| \leqslant 1$. Prove that

$$|a+b|^2 + |a-b|^2 \leqslant 4$$

Solution 1 by Michel Bataille, Rouen, France.

We have $|a+b|^2 = (a+b)(\overline{a}+\overline{b}) = |a|^2 + a\overline{b} + \overline{a}b + |b|^2$ and $|a-b|^2 = |a|^2 - a\overline{b} - \overline{a}b + |b|^2$, hence $|a+b|^2 + |a-b|^2 = 2(|a|^2+|b|^2)$. Thus, it suffices to show that $|a|^2+|b|^2 \leqslant 2$ or even that $|a| \leqslant 1$ and $|b| \leqslant 1$.
If $2|a|^2 \leqslant 1$, then certainly $|a| \leqslant 1$. If $2|a|^2 \geqslant 1$, then using the triangular inequality, we see that

$$1 \geqslant |a^4+1| = |(a^2+1)^2 - 2a^2| \geqslant \left| |a^2+1|^2 - 2|a|^2 \right| = 2|a|^2 - |a^2+1|^2$$

so that $2 \geqslant 1 + |a^2+1|^2 \geqslant 2|a^2| = 2|a|^2$ and $|a| \leqslant 1$ follows.
Similarly, we have $|b| \leqslant 1$ if $2|b|^3 = |2b^3| \leqslant 1$. If $|2b^3| \geqslant 1$, then, as above,

$$1 \geqslant |b^6+1| = |(b^3+1)^2 - 2b^3| \geqslant \left| |b^3+1|^2 - |2b^3| \right| = |2b^3| - |b^3+1|^2$$

hence $2 \geqslant 1 + |b^3+1|^2 \geqslant 2|b|^3$ and $|b| \leqslant 1$ follows.
In any case, we have $|a| \leqslant 1$ and $|b| \leqslant 1$.

Also solved by the problem proposer.

• **5771** *Proposed by Goran Conar, Varaždin, Croatia.*

Suppose $x_1, x_2, \ldots, x_n \geqslant e$. Prove

$$\frac{n}{7} > \sum_{j=1}^{n} \frac{x_j}{1+x_j^3} > \frac{n}{2}\left(\prod_{j=1}^{n} x_j\right)^{-2/n}.$$

Solution 1 by Devis Alvarado, Tegucigalpa, Honduras.

If $x \geqslant e$, then $x^2 - 7 > 0 \Rightarrow x^3 - 7x + 1 > 0 \Rightarrow \dfrac{x}{1+x^3} < \dfrac{1}{7}$,

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} < \sum_{j=1}^{n} \frac{1}{7} = \frac{n}{7}.$$

If $x \geqslant e \Rightarrow x^3 > 1 \Rightarrow 2x^3 > 1 + x^3 \Rightarrow \dfrac{2x}{1+x^3} > \dfrac{1}{x^2}$, then

$$\begin{aligned} \sum_{j=1}^{n} \frac{2x_j}{1+x_j^3} &> \sum_{j=1}^{n} \frac{1}{x_j^2} \\ &\geqslant n \sqrt[n]{\prod_{j=1}^{n} \left(\frac{1}{x_j^2}\right)} \\ &= n \left(\prod_{j=1}^{n} x_j\right)^{-\frac{2}{n}} \\ \Rightarrow \sum_{j=1}^{n} \frac{x_j}{1+x_j^3} &> \frac{n}{2} \left(\prod_{j=1}^{n} x_j\right)^{-\frac{2}{n}} \end{aligned}$$

Solution 2 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

For the leftmost inequality, note that because each $x_j^2 \geqslant e^2 \approx 7.389 > 7$,

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} < \sum_{j=1}^{n} \frac{x_j}{x_j^3} = \sum_{j=1}^{n} \frac{1}{x_j^2} < \sum_{j=1}^{n} \frac{1}{7} = \frac{n}{7}.$$

As for the rightmost inequality, first observe that

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} > \sum_{j=1}^{n} \frac{x_j}{x_j^3 + x_j^3} = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{x_j^2}.$$

Now apply the AM-GM inequality to conclude that

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} > \frac{1}{2} \sum_{x_j^{-2}} > \frac{n}{2} \sqrt[n]{\prod_{j=1}^{n} x_j^{-2}} = \frac{n}{2} \left(\prod_{j=1}^{n} x_j\right)^{-2/n}.$$

Solution 3 by Michel Bataille, Rouen, France.

Let $f(x) = \dfrac{x}{1+x^3}$. The derivative $f'(x) = \dfrac{1-2x^3}{(1+x^3)^2}$ is negative for $x \geqslant e$, hence the function f is strictly decreasing on $[e, \infty)$ and $f(x) \leqslant f(e) = \dfrac{e}{1+e^3} < \dfrac{1}{7}$ whenever $x \geqslant e$. In consequence

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} < n \cdot \frac{1}{7}$$

and the left inequality follows.
Now, let $g(x) = f(e^x)$ (for $x \geqslant 1$). A simple calculation shows that

$$g''(x) = e^x(1+e^{3x})^{-3}(4e^{6x} - 13e^{3x} + 1).$$

Since $e^{3x} \geqslant e^3 > \dfrac{13+\sqrt{153}}{8}$, we see that $g''(x) > 0$ for $x \geqslant 1$ and therefore g is convex on the interval $[1, \infty)$.
Since $\ln(x_j) \in [1, \infty)$ for $j = 1, \ldots, n$, Jensen's inequality yields

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} = \sum_{j=1}^{n} \frac{e^{\ln(x_j)}}{1+e^{3\ln(x_j)}} = \sum_{j=1}^{n} g(\ln(x_j)) \geqslant n \cdot \frac{e^{s/n}}{1+e^{3s/n}}$$

where $s = \sum_{j=1}^{n} \ln(x_j) = \ln(p)$ (setting $p = \prod_{j=1}^{n} x_j$). Thus, we have

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} \geqslant n \cdot \frac{p^{1/n}}{1+p^{3/n}} = nf(p^{1/n}).$$

But, for $x \geqslant e$ the inequality $f(x) > \dfrac{1}{2x^2}$ holds (as being equivalent to $x^3 > 1$), hence $f(p^{1/n}) > \dfrac{1}{2} \cdot p^{-2/n}$ so that

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} > \frac{n}{2} p^{-2/n}$$

(the right inequality) holds.

Solution 4 by Perfetti Paolo, Universitą di "Tor Vergata", Roma, Italy.

$$\left(\frac{x}{1+x^3}\right) => \frac{1}{2x^2} \iff x^3 \geqslant 1$$

hence

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} \geqslant \frac{1}{2}\sum_{j=1}^{n}\frac{1}{x^2} \underbrace{\geqslant}_{AGM} \frac{n}{2}\left(\prod_{j=1}^{n} x_j\right)^{-2/n}$$

L.h.s.

$$(1+x^3-7x)' = 3x^2-7 \geqslant 0 \iff x \geqslant \sqrt{7/3} = 2.\overline{3}$$

Hence a fortiori it is true for $x \geqslant e$. It follows

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^3} < \sum_{j=1}^{n}\frac{x_j}{7x_j} = \frac{n}{7}$$

The proof is completed.

Also solved by Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania and the problem proposer.

• **5772** *Proposed by by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

For the matrix A in $\mathcal{M}_2(\mathbb{R})$, solve the equation $A^3 = A - A^T$, where A^T denotes the transpose of A.

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$. Then

$$A^3 = \begin{pmatrix} a^3 + bc(2a+d) & b(a^2+ad+bc+d^2) \\ c(a^2+ad+bc+d^2) & d^3+bc(a+2d) \end{pmatrix}$$

and

$$A - A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}.$$

Equating entries of A^3 and $A - A^T$, we have

$$a^3 + bc(2a+d) = 0 = d^3 + bc(a+2d) \tag{1}$$

and

$$b(a^2+ad+bc+d^2) = b - c = -c(a^2+ad+bc+d^2). \tag{2}$$

From Equations (1), we see

$$a^3 - d^3 + bc(a-d) = 0$$
$$(a-d)(a^2 + ad + bc + d^2) = 0.$$

Meanwhile, from Equations (2), we get

$$(b+c)(a^2 + ad + bc + d^2) = 0.$$

Therefore, either $a^2 + ad + bc + d^2 = 0$ or $a - d = 0 = b + c$.

Case 1: If $a^2 + ad + bc + d^2 = 0$, then from Equations (2), we have $b - c = 0$. Notice that $\det\left(A - A^T\right) = (b-c)^2 = 0$, which means $0 = \det(A^3) = (ad - bc)^3$, and $ad = bc$. Thus,

$$0 = a^2 + ad + bc + d^2 = a^2 + 2ad + d^2 = (a+d)^2,$$

so that $d = -a$ and $0 = ad - bc = -a^2 - b^2$; thus, $a = b = c = d = 0$, and A is the zero matrix.

Case 2: If $a - d = 0 = b + c$, then $d = a$ and $c = -b$. Equations (1) and (2) simplify to $a^3 - b^2(3a) = 0$ and $b(3a^2 - b^2) = 2b$. Thus, $a(a^2 - 3b^2) = 0$ and $b(3a^2 - b^2 - 2) = 0$. If $a = 0$, then $b(-b^2 - 2) = -b(b^2 + 2) = 0$, so $b = 0$ and once again, A is the zero matrix. If $a \neq 0$, then $b \neq 0$, so that $a^2 = 3b^2$ and $2 = 3a^2 - b^2 = 8b^2$; thus $b = \pm\dfrac{1}{2}$ and $a = \pm\dfrac{\sqrt{3}}{2}$. Therefore, the complete set of solutions is given by

$$A \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \right\}.$$

We remark that the nonzero solutions for A are rotation matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where θ has a reference angle of 30°.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $A^3 = \begin{pmatrix} a^3 + 2abc + bcd & b(a^2 + ad + bc + d^2) \\ c(a^2 + ad + bc + d^2) & d^3 + 2bcd + abc \end{pmatrix}$, and $A - A^T = \begin{pmatrix} 0 & b - c \\ -(b-c) & 0 \end{pmatrix} = (b-c)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

So $A - A^T$ is a rotation matrix multiplied by the scale factor $b - c$, and therefore so is A^3, so that A is also a rotation matrix multiplied by a scale factor.

If $b - c = 0$ (i.e., $b = c$), then $A - A^T$ is just the zero matrix. With $A^3 = A - A^T$ being the zero matrix and A being a rotation matrix multiplied by a scale factor, A is also the zero matrix. So one solution to the equation is the zero matrix (which is also clear from inspection).

Assume now that $b - c \neq 0$. Note that since A is a rotation matrix multiplied by a scale factor, $d = a$. So $A^3 = \begin{pmatrix} a^3 + 2abc + bcd & b(a^2 + ad + bc + d^2) \\ c(a^2 + ad + bc + d^2) & d^3 + 2bcd + abc \end{pmatrix} = \begin{pmatrix} a^3 + 3abc & b(3a^2 + bc) \\ c(3a^2 + bc) & a^3 + 3abc \end{pmatrix}$. Equating the off-diagonal elements of A^3 and $A - A^T$ gives $b(3a^2 + bc) = b - c$ and $c(3a^2 + bc) = -(b - c)$. Dividing the former equation by the latter yields $\frac{b}{c} = \frac{b - c}{-(b - c)} = -1$, so that $c = -b$. So $A - A^T = (b - c)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A^3 = \begin{pmatrix} a^3 - 3ab^2 & b(3a^2 - b^2) \\ -b(3a^2 - b^2) & a^3 - 3ab^2 \end{pmatrix}$. Equating the upper-left elements of A^3 and $A - A^T$ yields $a^3 - 3ab^2 = 0$, whence $a(a^2 - 3b^2) = 0$, so that $a = 0$ or $a = \pm\sqrt{3}b$.

We first consider the $a = 0$ case. We have $d = a = 0$ so that $A = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A^3 = b^3\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -b^3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $A^3 = A - A^T = 2b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have $-b^3 = 2b$, that is, $b(b^2 - 2) = 0$, so that $b = 0$ or $b = \pm\sqrt{2}$. If $b = 0$, then A is the zero matrix. If $b = \pm\sqrt{2}$, then $A = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$. But

$$\begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}^3 = 2\begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \neq 2\begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}^T.$$

Similarly,

$$\begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}^3 = 2\begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} \neq 2\begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}^T.$$

So the zero matrix is the only solution to the equation $A^3 = A - A^T$ that we have obtained so far.

We now turn to the case when $a = \pm\sqrt{3}b$. If $a = \sqrt{3}b$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{3}b & b \\ -b & \sqrt{3}b \end{pmatrix} = b\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}.$$

So $A^3 = 8b^3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $A^3 = A - A^T = 2b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have $8b^3 = 2b$, whence $b(4b^2 - 1) = 0$, so that $b = 0$ or $b = \pm\frac{1}{2}$. If $b = 0$, then A is the zero matrix. If $b = \frac{1}{2}$, then

$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. This matrix indeed satisfies the equation:

$$A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = A - A^T.$$

With $a = \sqrt{3}b$ and $b = -\frac{1}{2}$, then $A = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ and

$$A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = A - A^T.$$

If $a = -\sqrt{3}b$, then

$$A = b \begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}.$$

So $A^3 = 8b^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As above, since $A^3 = A - A^T = 2b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we again have $8b^3 = 2b$ and hence again either $b = 0$ (the zero matrix case) or $b = \pm\frac{1}{2}$. With $a = -\sqrt{3}b$ and $b = \frac{1}{2}$, $A = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ and

$$A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = A - A^T.$$

With $a = -\sqrt{3}b$ and $b = -\frac{1}{2}$, $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ and

$$A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = A - A^T.$$

So the five solutions to the equation $A^3 = A - A^T$ are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}, \text{ and } \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Solution 3 by Michel Bataille, Rouen, France.

Let O_2 and I_2 denote the zero matrix and the unit matrix of $\mathcal{M}_2(\mathbb{R})$ and let $A_0 = \frac{1}{2} \cdot \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$. It is readily checked that O_2, A_0, $-A_0$, A_0^T, $-A_0^T$ are solutions. We will show that there are no other solutions.

Let A be an arbitrary solution and let t and δ denote its trace and its determinant, respectively. From the Hamilton-Cayley theorem (or directly) we have $A^2 = tA - \delta I_2$, hence

$$A - A^T = A^3 = tA^2 - \delta A = t(tA - \delta I_2) - \delta A = (t^2 - \delta)A - t\delta I_2.$$

From $(A^3)^T = A^T - A = -A^3$, we then deduce that $(t^2 - \delta)A^T - t\delta I_2 = -(t^2 - \delta)A + t\delta I_2$ so that $(t^2 - \delta)(A + A^T) = 2t\delta I_2$.
Suppose that $\delta = t^2$. Then, $\delta t = 0$, hence $t = \delta = 0$. It follows that $A^2 = O_2$, hence $A^3 = O_2$ and $A = A^T$. Therefore A must be of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $\delta = -a^2 - b^2$. Since $\delta = 0$, we see that $a = b = 0$. Thus $A = O_2$.
Suppose that $\delta \neq t^2$. From the expressions of $A + A^T$ and $A - A^T$ found earlier, we readily obtain

$$A(2 + \delta - t^2) = \frac{t\delta}{t^2 - \delta} \cdot (2 + \delta - t^2)I_2.$$

If $t^2 \neq \delta + 2$, then $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for some real a and $A^3 = A - A^T$ gives $a = 0$, hence $A = O_2$.

If $t^2 = \delta + 2$, then $A + A^T = t\delta I_2$, which, setting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, yields $a = d = \frac{t\delta}{2}$ and $b + c = 0$ so that $t = a + d = t\delta$. Note that $t \neq 0$ since otherwise $a = d = 0$ so that $\delta = b^2$, contradicting $\delta = -2$. Therefore $\delta = 1$, hence $t^2 = 3$ and $a = d = \frac{t}{2}$. Since $1 = \delta = \frac{t^2}{4} + b^2$, we obtain $b^2 = \frac{1}{4}$. Thus, $a = \pm\frac{\sqrt{3}}{2}$, $b = \pm\frac{1}{2}$ and A must be one of the four matrices A_0, $-A_0$, A_0^T, $-A_0^T$. This completes the proof.

Also solved by Yunyong Zhang, Chinaunicom, Yunnan, China; Srikanth Pai, Mudhitha Maths Academy, Bangalore, India and the problem proposer.

• **5773** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the following integral

$$I = \int_0^1 \frac{x \ln^2 x}{x^3 + x\sqrt{x} + 1} dx.$$

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $y = \sqrt{x}$, then $\mathrm{d}y = \dfrac{1}{2\sqrt{x}}\mathrm{d}x,\ \mathrm{d}x = 2\sqrt{x}\mathrm{d}y,\ x = y^2.$

$$I = \int_0^1 \frac{y^2 \ln^2 y^2}{y^6 + y^3 + 1} 2y\mathrm{d}y = 8\int_0^1 \frac{y^3 \ln^2 y}{1 + y^3 + y^6}\mathrm{d}y,$$

$$= 8\int_0^1 \frac{y^3(1 - y^3)\ln^2 y}{1 - y^9}\mathrm{d}y,$$

$$= 8\int_0^1 (y^3 - y^6)\ln^2 y \sum_{n=0}^{\infty} y^{9n}\mathrm{d}y,$$

$$= 8\sum_{n=0}^{\infty}\int_0^1 \left(y^{9n+3}\ln^2 y - y^{9n+6}\ln^2 y\right)\mathrm{d}y,$$

$$= 8\sum_{n=0}^{\infty}\left[\frac{2}{(9n+4)^3} - \frac{2}{(9n+7)^3}\right],$$

$$= 16\sum_{n=0}^{\infty}\left[\frac{1}{(9n+4)^3} - \frac{1}{(9n+7)^3}\right],$$

$$\approx 16 \times 0.012965 \approx 0.20744.$$

Solution 2 by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.

$$I = \int_0^1 \frac{x\ln^2(x)}{(x^{\frac{3}{2}})^2 + x^{\frac{3}{2}} + 1}dx$$

Multiplying numerator and denominator by $(1 - x^{\frac{3}{2}})$,

$$I = \int_0^1 \frac{x\ln^2(x)(1 - x^{\frac{3}{2}})}{(1 - x^{\frac{9}{2}})}dx$$

Since $|x^{\frac{9}{2}}| < 1$ as x goes from 0 to 1 , we can use infinite geometric series expansion,

$$I = \int_0^1 x\ln^2(x)(1 - x^{\frac{3}{2}})\sum_{n=0}^{\infty} x^{\frac{9n}{2}}dx$$

Taking constants inside the sum and interchanging sum and interval using dominated convergence theorem,

$$I = \sum_{n=0}^{\infty}\int_0^1 x\ln^2(x)(1 - x^{\frac{3}{2}})x^{\frac{9n}{2}}dx = \sum_{n=0}^{\infty}\int_0^1 \left(\ln^2(x)x^{\frac{9n}{2}+1} - \ln^2(x)x^{\frac{9n}{2}+\frac{5}{2}}\right)dx$$

Using $\int_0^1 \ln^n(x)x^m dx = (-1)^n\frac{\Gamma(n+1)}{(m+1)^{n+1}}$,

$$I = \sum_{n=0}^{\infty}\frac{\Gamma(3)}{(\frac{9n}{2}+2)^3} - \frac{\Gamma(3)}{(\frac{9n}{2}+\frac{7}{2})^3} = \frac{8}{729}\sum_{n=0}^{\infty}\frac{2}{(n+\frac{4}{9})^3} - \frac{2}{(n+\frac{7}{9})^3}$$

Using $\psi''(x) = \sum_{n=0}^{\infty}\frac{-2}{(n+x)^3}$,

$$I = \frac{8}{729}\left(\psi''(\frac{7}{9}) - \psi''(\frac{4}{9})\right) = 0.20744$$

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

Let's substitute x by $y^{\frac{2}{3}}$ where $\frac{dx}{dy} = \frac{2}{3}y^{-\frac{1}{3}}$, we get: $I = \left(\frac{2}{3}\right)^3\int\limits_0^1\frac{y^{\frac{1}{3}}\ln^2(y)}{y^2+y+1}dy$

By $y^2+y+1 = \frac{y^3-1}{y-1}$: $I = \left(\frac{2}{3}\right)^3\int\limits_0^1\frac{y^{\frac{4}{3}}\ln^2(y)}{y^3-1}dy - \left(\frac{2}{3}\right)^3\int\limits_0^1\frac{y^{\frac{1}{3}}\ln^2(y)}{y^3-1}dy$

Performing the following substitution: $y^3 = z$ and $\dfrac{dx}{dy} = \dfrac{1}{3}y^{-\frac{2}{3}}$

$$I = \left(\frac{2}{9}\right)^3 \int\limits_0^1 \frac{z^{-\frac{2}{9}} \ln^2(z)}{z-1}dz - \left(\frac{2}{9}\right)^3 \int\limits_0^1 \frac{z^{-\frac{5}{9}} \ln^2(z)}{z-1}dz$$

Known the integral representation of the m-ordered polygamma function:

$$\psi^{(m)}(z) = -\int\limits_0^1 \frac{t^{z-1} ln^m(z)}{1-t}dt$$

The result is the following:

$$I = \left(\frac{2}{9}\right)^3 \left[\psi^{(2)}(\frac{7}{9}) - \psi^{(2)}(\frac{4}{9})\right] \approx 0.2074404480426$$

was determined by WolframAlpha.

Solution 4 by Devis Alvarado, Tegucigalpa, Honduras.

$$\begin{aligned}
I &= \int_0^1 \frac{x\ln^2(x)}{x^3 + x\sqrt{x} + 1}dx \\
&= 8\int_0^\infty \frac{y^2 e^{-4y}}{e^{-6y} + e^{-3x} + 1}dy,\ y = -\frac{1}{2}\ln(x) \Rightarrow dx = -2e^{-2y}dy \\
&= 8\int_0^\infty \frac{y^2 e^{-4y}\left(1 - e^{-3y}\right)}{1 - e^{-9y}}dy \\
&= 8\int_0^\infty y^2 e^{-4y}\left(1 - e^{-3y}\right)\left(\sum_{k=0}^\infty e^{-9ky}\right)dy \\
&= 8\sum_{k=0}^\infty \int_0^\infty \left(y^2\left(1 - e^{-3y}\right)e^{-(9k+4)y}\right)dy \\
&= 8\sum_{k=0}^\infty \int_0^\infty \left(y^2 e^{-(9k+4)y} - y^2 e^{-(9k+7)y}\right)dy \\
&= 8\sum_{k=0}^\infty \left(\frac{2}{(9k+4)^3} - \frac{2}{(9k+7)^3}\right) \\
&= -\frac{8}{9^3}\psi^{(2)}\left(\frac{4}{9}\right) + \frac{8}{9^3}\psi^{(2)}\left(\frac{7}{9}\right) \\
&= \frac{8}{729}\left(\psi^{(2)}\left(\frac{7}{9}\right) - \psi^{(2)}\left(\frac{4}{9}\right)\right)
\end{aligned}$$

where $\psi^{(2)}(z)$ is funtion trigamma

$$\psi^{(2)}(z) = -\sum_{k=0}^{\infty} \frac{2}{(k+z)^3} = \frac{d^3}{dz^3} \ln(\Gamma(z)).$$

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

In the following solution, we use (1) the fact that $\int_0^1 x^a \ln^2 x\, dx = 2/(a+1)^3$, $a > -1$, and (2) the definition of the polygamma function, $\psi_2(x) = -2\sum_{n=0}^{\infty} 1/(n+x)^3$. We have

$$\begin{aligned}
I &\overset{x\mapsto y^{2/3}}{=} \frac{8}{27}\int_0^1 \frac{y^{1/3}\ln^2 y}{y^2+y+1}\,dy = \frac{8}{27}\int_0^1 \frac{(1-y)y^{1/3}\ln^2 y}{1-y^3}\,dy \\
&= \frac{8}{27}\left(\int_0^1 \frac{y^{1/3}\ln^2 y}{1-y^3}\,dy - \int_0^1 \frac{y^{4/3}\ln^2 y}{1-y^3}\,dy\right) \\
&= \frac{8}{27}\left(\int_0^1 \sum_{n=0}^{\infty} y^{3n+1/3}\ln^2 y\,dy - \int_0^1 \sum_{n=0}^{\infty} y^{3n+4/3}\ln^2 y\,dy\right) \\
&= \frac{8}{27}\left(\sum_{n=0}^{\infty}\int_0^1 y^{3n+1/3}\ln^2 y\,dy - \sum_{n=0}^{\infty}\int_0^1 y^{3n+4/3}\ln^2 y\,dy\right) \\
&\overset{(1)}{=} \frac{8}{27}\left(\sum_{n=0}^{\infty}\frac{2}{(3n+4/3)^3} - \sum_{n=0}^{\infty}\frac{2}{(3n+7/3)^3}\right) \\
&= \frac{8}{(27)^2}\left(\sum_{n=0}^{\infty}\frac{2}{(n+4/9)^3} - \sum_{n=0}^{\infty}\frac{2}{(n+7/9)^3}\right) \\
&\overset{(2)}{=} \frac{8}{(27)^2}\left(-\psi_2(4/9) + \psi_2(7/9)\right) \approx 0.20744.
\end{aligned}$$

Alternatively, we can write the solution in terms of Hurwitz's function:

$$I = \frac{16}{(27)^2}\left(\zeta(3,4/9) - \zeta(3,7/9)\right).$$

Solution 6 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

The substitution $x = t^2$ yields

$$I = 8\int_0^1 \frac{t^3\ln^2 t\,dt}{t^6+t^3+1} = 8\int_0^1 \frac{(t^3-1)t^3\ln^2 t\,dt}{(t^3-1)(t^6+t^3+1)} = 8\int_0^1 \frac{(-t^3+1)t^3\ln^2 t\,dt}{1-t^9} =$$

$$= 8\sum_{k=0}^{\infty}\int_0^1 (-t^3+1)t^3 t^{9k}\ln^2 t\,dt = 8\sum_{k=0}^{\infty}\int_0^1 (t^{9k+3} - t^{9k+6})\ln^2 t\,dt =$$

$$= 16\sum_{k=0}^{\infty}\left(\frac{1}{(9k+4)^3} - \frac{1}{(9k+7)^3}\right) = \frac{-16}{1458}\left(\Psi^{(2)}(\frac{4}{9}) - \Psi^{(2)}(\frac{7}{9})\right).$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo and the problem proposer.

• **5774** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

If $x \in \left[0, \pi/2\right]$, then show that

$$\left(\sin^2 x\right)^{\cos^2 x} + \left(\cos^2 x\right)^{\sin^2 x} \leqslant \frac{3}{2}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Noting that the inequality holds for $x = 0$ and $x = \pi/2$, we shall assume that $x \in (0, 1)$ and use Bernoulli's inequality:

$$(1 + x)^{\alpha} < 1 + \alpha x \quad \text{for all } \alpha \in (0, 1) \text{ and } x > -1, x \neq 0.$$

Setting $X = \sin^2 x$ and $\alpha = 1 - X$, we get

$$X^{1-X} = \big(1 + (X - 1)\big)^{1-X} < 1 + (1 - X)(X - 1) = 2X - X^2 \tag{3}$$

and

$$(1 - X)^X = (1 + (-X))^X < 1 + X(-X) = 1 - X^2. \tag{4}$$

Adding (1) and (2), we obtain

$$(\sin^2 x)^{\cos^2 x} + (\cos^2 x)^{\sin^2 x} = X^{1-X} + (1 - X)^X < 1 + 2X - 2X^2.$$

An elementary calculus computation yields $\max\limits_{0<X<1} (1 + 2X - 2X^2) = 3/2$, attained when $X = 1/2$, or $x = \pi/4$. However, the maximum value attained by the left-hand function is actually $2\left(\dfrac{1}{2}\right)^{1/2} = \sqrt{2} < 3/2$.

Solution 2 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

We will prove the stronger statement that the function $f(x) = \left(\sin^2 x\right)^{\cos^2 x} + \left(\cos^2 x\right)^{\sin^2 x}$ has a maximum value of $\sqrt{2}$ on the interval $[0, \pi/2]$, which is attained when $x = \pi/4$. First note that

$$f(x) = \left(\sin^2 x\right)^{1-\sin^2 x} + \left(\cos^2 x\right)^{1-\cos^2 x} = \left(\sin^2 x\right) g\left(\sin^2 x\right) + \left(\cos^2 x\right) g\left(\cos^2 x\right)$$

where $g(t) = t^{-t} = e^{-t\ln t}$, which is defined for $t \geqslant 0$. (For the sake of convenience, we define $g(0) := \lim_{t\to 0^+} g(t) = 1$.) Observe that

$$g'(t) = e^{-t\ln t}(-\ln t - 1) = -t^{-t}(\ln t + 1),$$

which is negative on $(1/e, \infty)$. Furthermore,

$$g''(t) = t^{-t}(\ln t + 1)^2 - t^{-t-1} = t^{-t-1}\left[t(\ln t + 1)^2 - 1\right].$$

Now if we set $h(t) = t(\ln t + 1)^2$, then

$$h'(t) = (\ln t + 1)^2 + 2(\ln t + 1) = (\ln t + 1)(\ln t + 3).$$

Hence h is increasing on $(0, 1/e^3) \cup (1/e, \infty)$ and decreasing on $(1/e^3, 1/e)$. Since $h(1/e^3) = 4/e^3$, $h(1/e) = 0$, and $h(1) = 1$, we can conclude that $h(t) \leqslant 1$ for $0 \leqslant t \leqslant 1$. Consequently, $g''(t) \leqslant 0$ on $[0, 1]$, which implies that g is concave down over this interval. Because the nonnegative weights $\sin^2 x$ and $\cos^2 x$ sum to 1, we may invoke Jensen's inequality to find that

$$f(x) \leqslant g\left(\sin^2 x \cdot \sin^2 x + \cos^2 x \cdot \cos^2 x\right) = \left(\sin^4 x + \cos^4 x\right)^{-\left(\sin^4 x + \cos^4 x\right)}.$$

Let us now consider one final function $F(x) = \sin^4 x + \cos^4 x$. Note that

$$F'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x = 4\sin x\cos x(\sin^2 x - \cos^2 x) = -4\sin x\cos x\cos(2x),$$

and so the critical numbers of $F(x)$ are $x = n\pi$, $x = (2n + 1)\pi/2$, and $x = (2n + 1)\pi/4$, where $n \in \mathbb{Z}$. Out of these, the only critical number in $(0, \pi/2)$ is $\pi/4$. Comparing $F(\pi/4) = 1/2$ to the endpoint arguments $F(0) = F(\pi/2) = 1$ shows that the minimum value of $F(x)$ is $1/2$ on the interval $[0, \pi/2]$. Yet $1/2 > 1/e$, and since g is decreasing on $(1/e, \infty)$, it follows that

$$f(x) \leqslant g\left(\sin^4 x + \cos^4 x\right) \leqslant g\left(\frac{1}{2}\right) = \sqrt{2}$$

for $0 \leqslant x \leqslant \pi/2$. Optimality thus follows from the fact that $f(\pi/4) = \sqrt{2}$ itself.

Solution 3 by Biao Zeng, Fort Hays State University, Hays, KS.

This is equivalent to showing that for $r \in [0, 1]$, prove that

$$r^{1-r} + (1 - r)^r \leqslant \frac{3}{2}$$

Let $f(r) = r^{1-r} + (1 - r)^r$, f is continuous on closed interval $[0, 1]$, so f has maximum value. Calculate derivative of f on (0, 1).

$$f'(r) = r^{1-r}\left(-lnr + \frac{1-r}{r}\right) + (1 - r)^r\left(ln(1 - r) - \frac{r}{1-r}\right)$$

Solve equation $f'(r) = 0$, we can see that $r = \dfrac{1}{2}$ is a solution. It is also easy to see that if $r > \dfrac{1}{2}, f' < 0$ and if $r < \dfrac{1}{2}, f' > 0$. So f achieve maximum when $r = \dfrac{1}{2}$. The maximum value of f is $\sqrt{2}$ which is less than $\dfrac{3}{2}$.

Solution 4 by Michel Bataille, Rouen, France.

Since $\sin^2 x + \cos^2 x = 1$, one of the two non-negative numbers $\sin^2 x$, $\cos^2 x$ is less than or equal to $\dfrac{1}{2}$, say $\sin^2 x \leqslant \dfrac{1}{2}$. For simplicity, let $a = \sin^2 x$. We have to prove that $a^{1-a} + (1-a)^a \leqslant \dfrac{3}{2}$.
Consider the function $f(x) = x^x = e^{x \ln x}$ (with $f(0) = 1$). We have $f'(x) = (1 + \ln x)x^x$ for $x \in (0, 1]$, hence the minimal value of f on $[0, 1/2]$ is $f(1/e) = e^{-1/e}$. In particular $a^a \geqslant e^{-1/e}$ and therefore $a^{1-a} = \dfrac{a}{a^a} \leqslant a \cdot e^{1/e}$.
On the other hand, we have $1 - a \in [1/2, 1]$ and

$$(1-a)^a = \frac{1-a}{(1-a)^{1-a}}.$$

On $[1/2, 1]$, f is increasing (since $\dfrac{1}{2} > \dfrac{1}{e}$), hence $f(1-a) \geqslant f(1/2) = e^{-(\ln 2)/2}$ and therefore $(1-a)^a \leqslant (1-a)e^{(\ln 2)/2} = (1-a)\sqrt{2}$.
As a result, we obtain

$$a^{1-a} + (1-a)^a \leqslant ae^{1/e} + (1-a)\sqrt{2} = \sqrt{2} + a(e^{1/e} - \sqrt{2}).$$

To conclude, we observe that $e^{1/e} - \sqrt{2} > 0$ so that $a^{1-a} + (1-a)^a \leqslant \sqrt{2} + \dfrac{e^{1/e} - \sqrt{2}}{2}$ and the required result follows since $\sqrt{2} + \dfrac{e^{1/e} - \sqrt{2}}{2} < \dfrac{3}{2}$.

Solution 5 by Perfetti Paolo, Universitą di “Tor Vergata", Roma, Italy.

Weighted AGM $x^a y^b \leqslant xa + yb$, $a, b \geqslant 0$, $a + b = 1$.

$$\sin^2 x \cdot (1 - \sin^2 x) + 1 \cdot \sin^2 x \geqslant \left(\sin^2 x\right)^{\cos^2 x} \cdot 1^{\sin^2 x} = \left(\sin^2 x\right)^{\cos^2 x}$$

$$\cos^2 x \cdot \sin^2 x + 1 \cdot (1 - \sin^2 x) \geqslant \left(\cos^2 x\right)^{\sin^2 x} \cdot 1^{1-\sin^2 x} = \left(\cos^2 x\right)^{\sin^2 x}$$

Hence

$$\left(\sin^2 x\right)^{\cos^2 x} + \left(\cos^2 x\right)^{\sin^2 x} \leqslant 2\sin^2 x \cos^2 x + 1 = \frac{(\sin(2x))^2}{2} + 1 \leqslant \frac{3}{2}$$

or $|\sin(2x)| \leqslant 1$ clearly true.

Solution 6 by Péter Fülöp, Gyömrő, Hungary.

Using the Half-angle formulaes we get:

$$LHS = \left(\frac{1-\cos(2x)}{2}\right)^{\frac{1+cos(2x)}{2}} + \left(\frac{1+\cos(2x)}{2}\right)^{\frac{1-cos(2x)}{2}}$$

Let's introduce $u = \dfrac{1+\cos(2x)}{2}$ notation:

$$LHS = f(u) = u^{1-u} + (1-u)^u$$

where $u \in [0,1]$

Note that the function $f(u)$ has a local maximum in $\in [0,1]$ domain:

It can be found by solving the $f'(u) = 0$ equation:

$$u^{1-u}\left(\frac{1-u}{u} - ln(u)\right) = (1-u)^u\left(\frac{u}{1-u} - ln(1-u)\right)$$

It can be realized the symmetry between u and $1-u$

The $f'(u) = 0$ equation is exist if $u = 1-u$ i.e. $u = \dfrac{1}{2}$

After performing the analysis of the function we get that at $u = \dfrac{1}{2}$ is a local maximum of the $f(u)$ function in $u \in [0,1]$ domain.

It means that $x = \dfrac{\pi}{4}$, put it back to the original inequalty:

$$LHS \leqslant \left(\frac{1}{2}\right)^{\frac{1}{2}} + \left(\frac{1}{2}\right)^{\frac{1}{2}} = \sqrt{2} < \frac{3}{2}$$

The inequality is proved with the exception of $LHS = \dfrac{3}{2}$.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Yunyong Zhang, Chinaunicom, Yunnan, China and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica ($\longleftarrow$ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣