# Problems and Solutions

Albert Natian, Section Editor
February 2025 Vol 125[1]

************************************************************

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

**Solutions to the problems published in this issue should be submitted *before* June 1, 2025.**

• **5799** *Proposed by Syed Shahabudeen, Ernakulam, Kerala, India.*

Prove that
$$\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})dx = G + \frac{1}{2}$$
where $G$ is the Catalan's Constant and $\mathbf{K}$ is the complete elliptic integral of the first kind defined as
$$\mathbf{K}(t) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - t^2 \sin^2\theta}}.$$

• **5800** *Proposed by Michael Brozinsky, Central Islip, New York.*

Show that in any group of 8 continuous functions, each having the set of real numbers as its domain, there exist (at least) 4 whose graphs mutually intersect or (at least) 3 whose graphs mutually do not intersect.

• **5801** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Solve for real $x$:
$$\frac{1}{1+x^4} + \frac{1}{2+x^6} + \frac{1}{3+x^8} + \frac{1}{4+x^{10}} = \frac{77}{60x^2}.$$

• **5802** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Let $a, b, c > 0$. Show that
$$\frac{1}{b^2}\sqrt{\frac{a^5+b^5}{a+b}} + \frac{1}{c^2}\sqrt{\frac{b^5+c^5}{b+c}} + \frac{1}{a^2}\sqrt{\frac{c^5+a^5}{c+a}} \geqslant 3.$$

• **5803** *Proposed by Michel Bataille, Rouen, France.*

Let $n$ be a positive integer. Prove that

$$\sum_{1\leqslant i\leqslant j\leqslant n}\frac{1}{ij}=\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2}.$$

• **5804** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$J:=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\arccos x}{\sqrt{4x^4-5x^2+1}}dx.$$

# *Solutions*

*To Formerly Published Problems*

• **5775** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\right].$$

**Solution 1 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.**

Let $S$ denote the sum. Then

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\left(\sum_{k=0}^{\infty}(-1)^k\frac{1}{n+k}\right)\right]\\
&= \sum_{n=1}^{\infty}\left[\int_0^1 x^{2n-2}\,dx-\left(\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{n+k-1}\,dx\right)\right]\\
&= \sum_{n=1}^{\infty}\left[\int_0^1 x^{2n-2}\,dx-\int_0^1 x^{n-1}\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)\,dx\right]\\
&= \sum_{n=1}^{\infty}\left[\frac{1}{2}\int_0^1 x^{n-\frac{3}{2}}\,dx-\int_0^1 x^{n-1}\left(\frac{1}{1+x}\right)\,dx\right]\\
&= \sum_{n=1}^{\infty}\int_0^1\left[\frac{1}{2}x^{n-\frac{3}{2}}-x^{n-1}\left(\frac{1}{1+x}\right)\right]\,dx
\end{aligned}$$

In each exchange of integral with sum, it can be verified that bounded or dominated convergence is fulfilled. For the next steps, we simplify the expression and the change in the variables: $x = y^2 \Rightarrow dx = 2ydy$. We observe that the integral has a discontinuity for n=1 before the change, but it is resolved by separating the first term from the rest, and then making the change in the variables, and consequently the issue disappears.

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\int_0^1 x^{n-1}\left[\frac{1+x-2\sqrt{x}}{2\sqrt{x}(1+x)}\right]dx \\
&= \int_0^1 \left[\frac{1+y^2-2y}{2y(1+y^2)}\right]2y\,dy + \sum_{n=2}^{\infty}\int_0^1 y^{2n-2}\left[\frac{1+y^2-2y}{2y(1+y^2)}\right]2y\,dy \\
&= \int_0^1 \left[\frac{1+y^2-2y}{1+y^2}\right]dy + \int_0^1 \frac{y^2}{1-y^2}\left[\frac{1+y^2-2y}{1+y^2}\right]dy \\
&= \int_0^1 \left[1-\frac{2y}{1+y^2}\right]dy + \int_0^1 \frac{y^2}{1+y}\left[\frac{1-y}{1+y^2}\right]dy \\
&= \int_0^1 \left[1-\frac{2y}{1+y^2}\right]dy + \int_0^1 \left[-1+\frac{1}{1+y}+\frac{y}{1+y^2}\right]dy \\
&= \int_0^1 \left[\frac{1}{1+y}-\frac{y}{1+y^2}\right]dy \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.**

We have

$$\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\dots\right)=\int_0^1\left(t^{2n-2}-\frac{t^{n-1}}{1+t}\right)dt.$$

Clearly,

$$\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\dots\right)=O\left(\frac{1}{n}\right).$$

Hence

$$\sum_{n=1}^{2N}\left(\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\dots\right)\right)=\int_0^1\left(\frac{1-t^{4N}}{1-t^2}-\frac{1-t^{2N}}{1-t^2}\right)dt=$$

$$=\int_0^1 t^{2N}\frac{1-t^{2N}}{1-t^2}dt=\int_0^1\left(t^{2N}+t^{2N+2}+\dots+t^{4N-2}\right)dt=\frac{1}{2N+1}+\frac{1}{2N+3}+\dots+\frac{1}{4N-1}=$$

$$=\sum_{k=1}^{2N}\frac{1}{2N+k}-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{N+k}=H_{4N}-H_{2N}-\frac{1}{2}\left(H_{2N}-H_N\right)=H_{4N}-\frac{3}{2}H_{2N}+\frac{1}{2}H_N.$$

It is known (see for instance https://en.wikipedia.org/wiki/Harmonic_number) that

$$H_n=\ln n+\gamma+O\left(\frac{1}{n}\right)$$

as n tends to infinity, where $\gamma$ is the Euler-Mascheroni constant. So

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\dots\right)\right)=\lim_{N\to\infty}\left(H_{4N}-\frac{3}{2}H_{2N}+\frac{1}{2}H_N\right)$$

$$=\lim_{N\to\infty}\left(\ln(4N)\ -\frac{3}{2}\ln(2N)\ +\frac{1}{2}\ln(N)\right)\ =\frac{1}{2}\ln 2\,.$$

**Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

Let $S$ be the proposed sum. Then

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\frac{1}{2n}-\left(\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\right]\\
&= \sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)-\sum_{n=1}^{\infty}\left[\frac{1}{2n}-\left(\frac{1}{n+1}-\frac{1}{n+2}+\cdots\right)\right]\\
&= \ln 2-\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+2}\right)+\sum_{n=1}^{\infty}\left[\frac{1}{2n+2}-\left(\frac{1}{n+2}-\frac{1}{n+3}+\cdots\right)\right]\\
&= \ln 2-\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n+4}\right)+\sum_{n=1}^{\infty}\left[\frac{1}{2n+4}-\left(\frac{1}{n+2}-\frac{1}{n+3}+\cdots\right)\right]\\
&= \cdots\\
&= \ln 2-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+\cdots\\
&= \ln 2-\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+\cdots\right)
\end{aligned}$$

from where $S\ =\ln 2-\dfrac{1}{2}\ln 2=\dfrac{1}{2}\ln 2\approx 0.3465735\ldots$

**Solution 4 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.**

The sum is $\ln\sqrt{2}\approx 0.34657$.

Let

$$a_n=\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right).$$

For each positive integer $n$, let $b_n=a_{2n-1}+a_{2n}$. Thus, $b_1=a_1+a_2$, $b_2=a_3+a_4$, $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}a_n$, and

$$\begin{aligned}
b_n &= \frac{1}{4n-3} - \left(\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1} - \cdots\right) + \frac{1}{4n-1} - \left(\frac{1}{2n} - \frac{1}{2n+1} + \frac{1}{2n+2} - \cdots\right)\\
&= \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n-1}\\
&= \frac{1}{4n-3} - \frac{1}{4n-2} - \left(\frac{1}{4n-2} - \frac{1}{4n-1}\right)\\
&= \frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n} - \left(\frac{1}{4n-2} - \frac{1}{4n-1} + \frac{1}{4n-1} - \frac{1}{4n}\right)\\
&= \frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n} - \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n &= \sum_{n=1}^{\infty}\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right)\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\\
&= \frac{\ln 2}{2} = \ln\sqrt{2}.
\end{aligned}$$

**Solution 5 by Michel Bataille, Rouen, France.**

Let $a_n = \frac{1}{2n-1} - \sum_{k=0}^{\infty}\frac{(-1)^k}{n+k}$. We show that $\sum_{n=1}^{\infty} a_n = \frac{\ln 2}{2}$.

For $N \geqslant 1$, let $S_N = \sum_{n=1}^{N} a_n$. From the well-known $\sum_{k=0}^{\infty}\frac{(-1)^k}{n+k} = \int_0^1 \frac{u^{n-1}}{1+u}\,du$, we deduce

$$S_N = \sum_{n=1}^{N}\frac{1}{2n-1} - \int_0^1\left(\sum_{n=1}^{N}\frac{u^{n-1}}{1+u}\right)du = \sum_{n=1}^{N}\frac{1}{2n-1} - \int_0^1\frac{1-u^N}{1-u^2}\,du.$$

Note that

$$S_{N+1} - S_N = \frac{1}{2N+1} - \int_0^1\frac{u^N - u^{N+1}}{1-u^2}\,du = \frac{1}{2N+1} - \int_0^1\frac{u^N}{1+u}\,du$$

so that $\lim_{N\to\infty}(S_{N+1} - S_N) = 0$ (since $0 \leqslant \int_0^1 \frac{u^N}{1+u} \leqslant \int_0^1 u^N\,du = \frac{1}{N+1}$). In consequence, for the calculation of the limit of $S_N$ as $N \to \infty$, we may consider only $S_{2N}$. We obtain

$$\begin{aligned} S_{2N} &= \sum_{n=1}^{2N} \frac{1}{2n-1} - \int_0^1 \frac{1-u^{2N}}{1-u^2}\, du \\ &= \sum_{n=1}^{4N} \frac{1}{n} - \frac{1}{2}\sum_{n=1}^{2N} \frac{1}{n} - \int_0^1 (1 + u^2 + \cdots + u^{2N-2})\, du \\ &= H_{4N} - \frac{H_{2N}}{2} - (H_{2N} - \frac{H_N}{2}) = H_{4N} - \frac{3}{2}H_{2N} + \frac{1}{2}H_N \end{aligned}$$

where $H_m$ denote the $m$th harmonic number.
Since $H_m = \ln m + \gamma + o(1)$ as $m \to \infty$ (where $\gamma$ is Euler's constant), we see that

$$S_{2N} = 2\ln 2 + \ln N + \gamma - \frac{3}{2}(\ln 2 + \ln N + \gamma) + \frac{1}{2}(\ln N + \gamma) + o(1) \text{ as } N \to \infty.$$

Thus, $S_{2N} = \dfrac{\ln 2}{2} + o(1)$ as $N \to \infty$ and therefore $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_{2N} = \frac{\ln 2}{2}$.

**Solution 6 by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.**

We know that

$$\sum_{n=1}^{p} \frac{1}{2n-1} = H_{2p} - \frac{1}{2}H_p = \ln(2p) + \gamma - \frac{1}{2}\ln p - \frac{\gamma}{2} + o(1) = \ln 2 + \frac{1}{2}\ln p + \frac{\gamma}{2} + o(1)$$

$$\begin{aligned} &\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{n+k} = (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k} \doteq (-1)^n B_n = \\ &= \sum_{k=0}^{\infty} (-1)^k \int_0^1 t^{n+k-1} dt = \\ &= \int_0^1 \frac{t^{n-1}dt}{1+t} = \frac{t^n}{n(1+t)}\Big|_0^1 + \int_0^1 \frac{t^n dt}{n(1+t)^2} = \frac{1}{2n} + \frac{1}{4n(n+1)} + \\ &+ \int_0^1 \frac{2t^{n+1}dt}{n(n+1)(1+t)^3} = \frac{1}{2n} + \frac{1}{4n(n+1)} + \frac{1}{4n(n+1)(n+2)} + \\ &+ \int_0^1 \frac{6t^{n+2}}{4n(n+1)(n+2)(1+t)^4} = \frac{1}{2n} + \frac{1}{4n^2} + O(\frac{1}{n^4}) \end{aligned}$$

where

$$0 \leqslant \int_0^1 \frac{2t^{n+1}dt}{n(n+1)(n+2)(1+t)^4} \leqslant \int_0^1 \frac{3t^{n+1}dt}{2n(n+1)(n+2)} = \frac{3}{2n(n+1)(n+2)^2}$$

Let $S$ be the series.

$$S = \ln 2 + \frac{\gamma}{2} + \lim_{N\to\infty} \frac{1}{2}\ln N - \sum_{n=1}^{N} (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k} \qquad (1)$$

$\sum_{n=1}^{N}(-1)^n = -\frac{1-(-1)^N}{2} \doteq A_N$, $A_0 = 0$. Now let's use Abel's "Summation by parts".

$$\sum_{n=1}^{N}(-1)^n \sum_{k=n}^{\infty}\frac{(-1)^k}{k} =$$

$$-B_1 + \sum_{n=2}^{N}(-1)^n\sum_{k=n}^{\infty}\frac{(-1)^k}{k} = \underbrace{-B_1 - A_1B_1}_{=0} + \underbrace{A_NB_N}_{\to 0} + \sum_{n=2}^{N}A_{n-1}(B_{n-1}-B_n) =$$

$$= \sum_{n=2}^{N}A_{n-1}(B_{n-1}-B_n) = -\sum_{n=2}^{N}\frac{1+(-1)^n}{2}\frac{(-1)^{n-1}}{n-1} = \frac{-1}{2}\sum_{n=1}^{N-1}\frac{(-1)^n}{n} + \frac{1}{2}\sum_{n=1}^{N-1}\frac{1}{n} =$$

$$= \frac{\ln 2}{2} + \frac{\ln(N-1)}{2} + \frac{\gamma}{2} + o(1)$$

By inserting in (1) we get

$$S = \ln 2 + \frac{\gamma}{2} + \lim_{N\to\infty}\frac{1}{2}\ln N - \frac{\ln 2}{2} - \frac{\ln(N-1)}{2} - \frac{\gamma}{2} + o(1) = \frac{\ln 2}{2}$$

**Also solved by Mingcan Fan, Huizhou University, Huizhou, China; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Péter Fülöp, Gyömrő, Hungary and the problem proposer.**

• **5776** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy .*

Let $p$ a positive real number and let $\{a_n\}_{n\geqslant 1}$ be a sequence defined by $a_1 = 1$, $a_{n+1} = \frac{a_n}{1+a_n^p}$. Find those real values $q > 0$, such that the following series converges

$$\sum_{n=1}^{\infty}\left|a_n - (pn)^{-\frac{1}{p}} + \frac{p-1}{2p^2}\frac{\ln n}{n^{1+1/p}}\right|^q.$$

**Solution 1 by Michel Bataille, Rouen, France.**

By an easy induction, we obtain $a_n > 0$ for all $n \in \mathbb{N}$. It follows that $a_{n+1} < a_n$ for all $n \in \mathbb{N}$ so that $\{a_n\}_{n\geqslant 1}$ is decreasing and bounded below, hence convergent. Its limit $\ell$ satisfies $\ell = \frac{\ell}{1+\ell^p}$, hence $\ell = 0$.

If $p = 1$, then $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ (by induction) and so $\left|a_n - (pn)^{-\frac{1}{p}} + \frac{p-1}{2p^2}\frac{\ln n}{n^{1+1/p}}\right|^q$ vanishes for all $n \geqslant 1$ and all $q > 0$. Thus, for $p = 1$ the given series is convergent for all $q > 0$. Now, we suppose that $p \neq 1$. Let $f(x) = \frac{x}{1+x^p}$ and let $b_n = a_n^{-p}$.

Since $f(x) = x(1 - x^p + x^{2p} + o(x^{2p}))$ as $x \to 0^+$, we have, as $n \to \infty$,

$$\begin{aligned} b_{n+1} &= (f(a_n))^{-p} = (a_n - a_n^{p+1} + a_n^{2p+1} + o(a_n^{2p+1}))^{-p} \\ &= a_n^{-p}(1 + pa_n^p - pa_n^{2p} + \frac{p(p+1)}{2} a_n^{2p} + o(a_n^{2p})) \\ &= b_n + p + \frac{p(p-1)}{2} a_n^p + o(a_n^p). \end{aligned}$$

We deduce $b_{n+1} - b_n \sim p$ and $b_{n+1} - b_n - p \sim \dfrac{p(p-1)}{2} a_n^p$.
From $b_{n+1} - b_n \sim p$, we then deduce $b_n \sim np$ (by Stolz's Theorem), that is, $a_n^p \sim \dfrac{1}{np}$ as $n \to \infty$. It follows that $b_{n+1} - b_n - p \sim \dfrac{p-1}{2} \cdot \dfrac{1}{n}$ and a second application of Stolz's Theorem gives $b_n - np \sim \dfrac{p-1}{2} \sum_{k=1}^{n} \dfrac{1}{k} \sim \dfrac{p-1}{2} \ln(n)$. It follows that

$$\begin{aligned} a_n = b_n^{-1/p} &= \left(np + \frac{p-1}{2} \ln(n) + o(\ln(n))\right)^{-1/p} \\ &= (np)^{-1/p} \left(1 - \frac{p-1}{2p^2} \cdot \frac{\ln(n)}{n} + o((\ln(n))/n)\right) \\ &= (np)^{-1/p} - \frac{p-1}{2p^{2+\frac{1}{p}}} \cdot \frac{\ln(n)}{n^{1+\frac{1}{p}}} + o((\ln(n))/n^{1+\frac{1}{p}}). \end{aligned}$$

As a result, as $n \to \infty$

$$\left| a_n - (pn)^{-\frac{1}{p}} + \frac{p-1}{2p^2} \frac{\ln n}{n^{1+1/p}} \right|^q \sim |\alpha| \frac{(\ln(n))^q}{n^{q(1+\frac{1}{p})}}$$

where $\alpha = \left| \dfrac{p-1}{2p^2} (1 - p^{-1/p}) \right|^q$.
From known results about Bertrand's series, we deduce that if $p \neq 1$, the proposed series is convergent if and only if $q(1 + \dfrac{1}{p}) > 1$ i.e. $q > \dfrac{p}{p+1}$.

**Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.**

• **5777** *Proposed by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

Consider the sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ defined by the exponential generating function

$$\frac{1}{(1-x)e^t + x} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.$$

Show that
$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)\,dx = (-1)^n.$$

**Solution 1 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.**

We observe that

1. The function at $f(t,x) = \dfrac{1}{(1-x)e^t + x}$ at $t$ is the generating function of $P_n(x)$, so
$$P_n(x) = \frac{\partial^n}{\partial t^n} f(t,x)\bigg|_{t=0}$$

2. For infinitely often differentiable functions $g$ and $h$, it can be shown, by induction, that
$$\frac{d^n}{dt^n}\big(g(t)g(t)\big) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}\left[g(t)\right]\frac{d^{n-k}}{dt^{n-k}}\left[h(t)\right].$$

Thus
$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)dx = \int_0^1\left[\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x)\right]dx \\
&= \int_0^1\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^k}{\partial t^k}f(t,x)\bigg|_{t=0}\cdot\frac{\partial^{n-k}}{\partial t^{n-k}}f(t,x)\bigg|_{t=0}\right]dx \\
&= \int_0^1\left[\frac{\partial^n}{\partial t^n}\big(f(t,x)\big)^2\right]\bigg|_{t=0}dx = \frac{d^n}{dt^n}\left[\int_0^1\big(f(t,x)\big)^2\,dx\right]\bigg|_{t=0} \\
&= \frac{d^n}{dt^n}\left[\int_0^1\frac{1}{\big((1-x)e^x+x\big)^2}dx\right]\bigg|_{t=0} \\
&= \frac{d^n}{dt^n}\left[-\frac{1}{(1-e^t)\big((1-x)e^x+x\big)}\bigg|_0^1\right]\bigg|_{t=0} \\
&= \frac{d^n}{dt^n}\left[-\frac{1}{(1-e^t)}+\frac{1}{(1-e^t)\,e^t}\right]\bigg|_{t=0} \\
&= \frac{d^n}{dt^n}\left[e^{-t}\right]\bigg|_{t=0} = (-1)^n.
\end{aligned}$$

**Solution 2 by Péter Fülöp, Gyömrő, Hungary.**

Square the sum $\displaystyle\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}$ and express the resulting Cauchy product:

$$\left[\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}\right]^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_k(x)\frac{t^k}{k!}P_{n-k}(x)\frac{t^{n-k}}{(n-k)!}\ .$$

Based on the generating function, we have:

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n} P_k(x)P_{n-k}(x)\underbrace{\frac{n!}{(n-k)!k!}}_{\binom{n}{k}} = \left[\frac{1}{(1-x)e^t+x}\right]^2.$$

Integrate both sides on the interval $[0,1]$:

$$LHS = \int_0^1 \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x)dx.$$

Swap the order of integration and summations:

$$LHS = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)dx\ .$$

Performing the integration on the RHS:

$$RHS = \int_0^1\left(\frac{1}{(1-x)e^t+x}\right)^2 dx = \int_0^1\left(\frac{e^{-t}}{1+x\frac{1-e^t}{e^t}}\right)^2.$$

By the $u = x\dfrac{1-e^t}{e^t}$ substitution, we get:

$$RHS = \frac{e^{-t}}{1-e^t}\int_0^{\frac{1-e^t}{e^t}}\frac{1}{(1+u)^2}du = \frac{e^{-t}}{1-e^t}\left|\frac{u}{u+1}\right|_0^{\frac{1-e^t}{e^t}} = e^{-t},$$

$$RHS = e^{-t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}(-1)^n.$$

Comparing LHS and RHS expressions, we see that the equality holds if

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)dx = (-1)^n.$$

**Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

$\dfrac{1}{(1-x)e^t+x} = \displaystyle\sum_{n=0}^{\infty} \dfrac{P_n(x)}{n!} t^n$, so $\left\{\dfrac{P_n(x)}{n!}\right\}_{n=0}^{\infty}$ are the coefficients of the ordinary generating function of $\dfrac{1}{(1-x)e^t+x}$. The problem follows by the Cauchy product of function $\dfrac{1}{(1-x)e^t+x}$ by itself if $\displaystyle\int_0^1 \sum_{k=0}^{n} \dfrac{P_k(x)}{k!} \cdot \dfrac{P_{n-k}(x)}{(n-k)!}\, dx = \dfrac{(-1)^n}{n!}$. Since $e^{-t} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!} t^n$, it is enough to prove that $\displaystyle\int_0^1 \left(\dfrac{1}{(1-x)e^t+x}\right)^2 dx = e^{-t}$:

$$\begin{aligned}
\int_0^1 \left(\frac{1}{(1-x)e^t+x}\right)^2 dx &= \int_0^1 \frac{1}{\big((1-e^t)x+e^t\big)^2}\, dx \\
&= \frac{1}{1-e^t} \int_0^1 \frac{1-e^t}{\big((1-e^t)x+e^t\big)^2}\, dx \\
&= \left.\frac{-1}{1-e^t} \cdot \frac{1}{\big((1-e^t)x+e^t\big)}\right]_0^1 \\
&= \frac{-1}{1-e^t}\left(1-\frac{1}{e^t}\right) \\
&= e^{-t}.
\end{aligned}$$

**Solution 4 by Albert Stadler, Herrliberg, Switzerland.**

We have

$$\int_0^1 \left(\frac{1}{(1-x)\,e^t+x}\right)^2 dx = \int_0^1 \sum_{j=0}^{\infty} P_j(x)\,\frac{t^j}{j!} \bullet \sum_{k=0}^{\infty} P_k(x)\,\frac{t^k}{k!} dx =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \sum_{j,k\geqslant 0, j+k=n} \frac{n!}{j!k!} \int_0^1 P_j(x)\, P_k(x)\, dx \right) =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \sum_{k=0}^{n} \binom{n}{k} \int_0^1 P_k(x)\, P_{n-k}(x)\, dx \right).$$

Let a>1. Then

$$\int \left(\frac{1}{(1-x)\,a+x}\right)^2 dx = \frac{1}{(a-1)\big((1-x)\,a+x\big)} + C$$

which implies

$$\int_0^1 \left(\frac{1}{(1-x)\,a+x}\right)^2 dx = \frac{1}{(a-1)} - \frac{1}{(a-1)\,a} = \frac{1}{a}.$$

We conclude that

$$\int_0^1 \left(\frac{1}{(1-x)\,e^t+x}\right)^2 dx = e^{-t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k\,(x)\,P_{n-k}\,(x)\,dx\right).$$

Comparing coefficients of t$^n$ we find that $\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k\,(x)\,P_{n-k}\,(x)\,dx = (-1)^n$.

**Solution 5 by Michel Bataille, Rouen, France.**

For $x > 1$ and $t < \ln\left(\frac{x}{x-1}\right)$, we have $0 < \frac{(x-1)e^t}{x} < 1$, hence

$$\frac{1}{(1-x)e^t+x} = \frac{1}{x}\cdot\frac{1}{1-\frac{(x-1)e^t}{x}} = \frac{1}{x}\left(1+\frac{x-1}{x}e^t+\frac{(x-1)^2}{x^2}e^{2t}+\cdots\right).$$

Expanding the exponentials on the right, the coefficient of $\frac{t^n}{n!}$ is, if $n = 0$,

$$\frac{1}{x}\left(1+\frac{x-1}{x}+\frac{(x-1)^2}{x^2}+\cdots\right) = \frac{1}{x}\cdot\frac{1}{1-\frac{x-1}{x}} = 1$$

and, if $n \geqslant 1$:

$$\frac{1}{x}\sum_{k=1}^{\infty}k^n\left(\frac{x-1}{x}\right)^k = \frac{1}{x}\cdot\frac{1}{1-\frac{x-1}{x}}\,\omega_n\left(\frac{x-1}{x}\cdot\frac{1}{1-\frac{x-1}{x}}\right) = \omega_n(x-1)$$

where $\omega_n$ denotes degree $n$-geometric polynomial (see **[1]** for example).

Thus, for all $n = 0, 1, \cdots$, $P_n(x) = \omega_n(x-1) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}k!(x-1)^k$ where $\left\{{n \atop k}\right\}$ is the Stirling number of the second kind. Multiplying the defining series by itself, we obtain

$$\frac{1}{((1-x)e^t+x)^2} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}P_k(x)\frac{t^k}{k!}\cdot P_{n-k}(x)\frac{t^{n-k}}{(n-k)!}\right) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\cdot\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x).$$

Let $I_n = \sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)\,dx$. By integration and exchanging sum and integral (see a justification at the end), we obtain that for $|t| < \ln 2$,

$$\sum_{n=1}^{\infty}I_n\frac{t^n}{n!} = \int_0^1\frac{dx}{((1-x)e^t+x)^2} = \frac{1}{e^t-1}\left[\frac{1}{x(1-e^t)+e^t}\right]_{x=0}^{x=1} = e^{-t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}$$

(the calculation assumes $t \neq 0$, but the result is still valid if $t = 0$). We deduce that $I_n = (-1)^n$ for $n = 0, 1, 2, \ldots$.

*Justification of the exchange* $\sum / \int$:

It is enough to show that for every $n$,

$$\left|\frac{t^n}{n!}\cdot\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x)\right| < a_n$$

where $a_n$ is independent of $x \in [0,1]$ and $\sum_{n=1}^{\infty} a_n$ is convergent.

If $m$ is a nonnegative integer and $0 \leqslant x \leqslant 1$, we have

$$|P_m(x)| = \left|\sum_{k=0}^{m} k!\left\{{n \atop k}\right\}(x-1)^k\right| \leqslant \sum_{k=0}^{m} k!\left\{{n \atop k}\right\}|x-1|^k \leqslant \sum_{k=0}^{m} k!\left\{{n \atop k}\right\} = f_m$$

where $f_m$ denote the $m$th Fubini number. The inequality $f_m \leqslant \dfrac{m!}{(\ln 2)^{m+1}}$ for $m \geqslant 0$ is known (see **[2]**) and gives

$$\binom{n}{k}f_k f_{n-k} \leqslant \frac{n!}{(\ln 2)^{n+2}}, \quad (0 \leqslant k \leqslant n),$$

from which we deduce

$$\left|\frac{t^n}{n!}\cdot\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x)\right| \leqslant \frac{(n+1)|t|^n}{(\ln 2)^{n+2}}.$$

Since $\sum_{n=1}^{\infty}\dfrac{(n+1)|t|^n}{(\ln 2)^{n+2}} < \infty$ when $|t| < \ln 2$, we are done.


**[1]** K.N. Boyadzhiev, A Series Transformation Formula and Related Polynomials, *Int. J. of Math. and Math. Sci.* (2005) p. 3849-66

**[2]** Qing Zou, The Log-convexity of the Fubini Numbers, *Transactions on Combinatorics*, Vol. 7, No 2 (2018) p. 17-33.


**Solution 6 by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.**

$$P_k(x) = \frac{d^k}{dt^k}\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}\Big|_{t=0} = \frac{d^k}{dt^k}\frac{1}{x+e^t(1-x)}\Big|_{t=0}.$$

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)dx =$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}\frac{d^{n-k}}{ds^{n-k}}\left[\int_0^1 \frac{1}{x+e^t(1-x)}\frac{1}{x+e^s(1-x)}dx\right]\Big|_{t=0,s=0}=$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}\frac{d^{n-k}}{ds^{n-k}}\left[\int_0^1 \left(\frac{1-e^t}{x(1-e^t)+e^t}-\frac{1-e^s}{x(1-e^s)+e^s}\right)\frac{1}{e^s-e^t}dx\right]\Big|_{t=0,s=0}=$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}\frac{d^{n-k}}{ds^{n-k}}\left[\frac{1}{e^s-e^t}\ln\frac{x(1-e^t)+e^t}{x(1-e^s)+e^s}\Big|_0^1\right]\Big|_{t=0,s=0}=$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{ds^k}\frac{d^{n-k}}{dt^{n-k}}\left[\frac{s-t}{e^s-e^t}\right]\Big|_{t=0,s=0}. \qquad (1)$$

Let $t > s$.

$$\frac{d^k}{ds^k}\left[\frac{t-s}{e^t-e^s}\right]\Big|_{s=0}= \frac{d^k}{ds^k}\left[e^{-t}(t-s)\sum_{p=0}^{\infty}e^{(s-t)p}\right]\Big|_{s=0}$$

$$= e^{-t}\sum_{i=0}^{k}\binom{k}{i}(t-s)^{(i)}\left[\sum_{p=0}^{\infty}e^{(s-t)p}\right]^{(k-i)}\Big|_{s=0}$$

$$= e^{-t}t\sum_{p=0}^{\infty}p^k e^{-tp} - ke^{-t}\sum_{p=0}^{\infty}p^{k-1}e^{-tp} = t\sum_{p=0}^{\infty}p^k e^{-t(p+1)} - k\sum_{p=0}^{\infty}p^{k-1}e^{-t(p+1)}.$$

The derivative

$$\frac{d^{n-k}}{dt^{n-k}}\frac{d^k}{ds^k}\left[\frac{t-s}{e^t-e^s}\right]\Big|_{t=0,s=0}= \sum_{q=0}^{n-k}\binom{n-k}{q}t^{(q)}\left[\sum_{p=0}^{\infty}p^k e^{-t(p+1)}\right]^{(n-k-q)} +$$

$$-k(-1)^{n-k}\sum_{p=0}^{\infty}p^{k-1}(p+1)^{n-k}e^{-t(p+1)}$$

is

$$\lim_{t\to 0} t(-1)^{n-k}\sum_{p=0}^{\infty}p^k(p+1)^{n-k}e^{-t(p+1)} +$$

$$+\lim_{t\to 0}(n-k)(-1)^{n-k-1}\sum_{p=0}^{\infty}p^k(p+1)^{n-k-1}e^{-t(p+1)} +$$

$$-k(-1)^{n-k}\sum_{p=0}^{\infty}(p+1)^{n-k}p^{k-1}e^{-t(p+1)}.$$

The sum over $k$ in (1) yields

$$\lim_{t\to 0} t(-1)^n \sum_{p=0}^{\infty} e^{-t(p+1)} + \lim_{t\to 0}\left((-1)^{n-1}\sum_{p=0}^{\infty} ne^{-t(p+1)} + n(-1)^n \sum_{p=0}^{\infty} e^{-t(p+1)}\right) =$$

$$= \lim_{t\to 0} t(-1)^n \sum_{p=0}^{\infty} e^{-t(p+1)} = (-1)^n \lim_{t\to 0} \frac{t}{1-e^{-t}} = (-1)^n.$$

**Solution 7 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.**

Define the function

$$f_x(t) = \frac{1}{(1-x)e^t + x}.$$

Then $f_x^{(n)}(0) = P_n(x)$. By the product rule for differentiation, we obtain

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(x) &= \sum_{k=0}^{n} \binom{n}{k} f_x^{(k)}(0) f_x^{(n-k)}(0) = \left(f_x^2\right)^{(n)}(0) \\ &= \left(\frac{\partial}{\partial t}\right)^n \left(\frac{1}{(1-x)e^t + x}\right)^2 \Bigg|_{t=0}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} \int_0^1 P_k(x) P_{n-k}(x)\, dx \\ &= \left(\frac{\partial}{\partial t}\right)^n \int_0^1 \left(\frac{1}{(1-x)e^t + x}\right)^2 dx \Bigg|_{t=0} \\ &= \left(\frac{\partial}{\partial t}\right)^n e^{-t}\Bigg|_{t=0} = (-1)^n,\end{aligned}$$

since

$$\int_0^1 \left(\frac{1}{(1-x)e^t + x}\right)^2 dx = \frac{1}{e^t - 1}\frac{1}{(1-x)e^t + x}\Bigg|_{x=0}^{1} = \frac{1}{e^t - 1}\left(1 - e^{-t}\right) = e^{-t}.$$

**Also solved by and the problem proposer.**

• **5778** *Proposed by Narendra Bhandari, Bajura, Nepal.*

Prove

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+n)^2}\left(1 - (-1)^n + (-1)^{n-m}\left(3 + \frac{m-3n}{n+m} + \frac{n^2}{(m+n)^2}\right)\right) = G,$$

where $G$ is the Catalan's constant, which is defined as $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$.

**Solution 1 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepall; and Prakash Pant, The University of Vermont, Bardiya, Nepal.**

Simplify an expression in the double sum and write,

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+n)^2} \left( 1 - (-1)^n + (-1)^{n-m} \frac{(2m+n)^2}{(m+n)^2} \right).$$

Now re-write the latter as the sum of two convergent infinite sums

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+n)^2} \left(1 - (-1)^n\right) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{(m+n)^2} \right).$$

Since $1 - (-1)^n = 0$ for even $n$, then the latter can be simplified to

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^m(1-(-1))}{(2m+2n-1)^2} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{(m+n)^2} \right).$$

Interchanging the order of summation in the first double sum, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{2(-1)^m}{(2m+2n-1)^2} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{(m+n)^2} \right) =$$

$$= \sum_{n=1}^{\infty} \left[ \frac{-2}{(2n+1)^2} + \frac{2}{(2n+3)^2} - \frac{2}{(2n+5)^2} + \frac{2}{(2n+7)^2} \cdots \right]$$

$$+ \sum_{m=1}^{\infty} \left[ \frac{1}{m^2} - \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} - \frac{1}{(m+3)^2} \cdots \right]$$

which we re-write as

$$= \frac{-2}{3^2} + \frac{-2}{7^2} + \frac{-2}{11^2} + \frac{-2}{15^2} + \frac{-2}{19^2} \cdots$$

$$+ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

which, by pairing fractions with identical denominators, can be written as:

$$\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{15^2} + \cdots = G.$$

**Solution 2 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.**

Let $S$ denote the expression on the left side of the given equation. It's easy to see that

$$3+\frac{m-3n}{n+m}+\frac{n^2}{(n+m)^2}=\frac{(2m+n)^2}{(n+m)^2}.$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{(2m+n)^2}\left(1-(-1)^n+(-1)^{n-m}\frac{(2m+n)^2}{(n+m)^2}\right)=\\
=\ &\sum_{k=0}^{\infty}\frac{1}{(2m+2k)^2}\left((-1)^m\frac{(2m+2k)^2}{(2k+m)^2}\right)\\
&+\sum_{k=0}^{\infty}\frac{1}{(2m+2k+1)^2}\left(2-(-1)^m\frac{(2m+2k+1)^2}{(2k+1+m)^2}\right)=\\
=\ &\sum_{k=0}^{\infty}\left[\frac{2}{(2m+2k+1)^2}+(-1)^m\left(\frac{1}{(m+2k)^2}-\frac{1}{(m+2k+1)^2}\right)\right].
\end{aligned}$$

Using the above results, we have for the sum $S$:

$$\begin{aligned}
S &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n+(-1)^{n-m}\frac{(2m+n)^2}{(n+m)^2}\right)\\
&= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{2(-1)^m}{(2m+2k+1)^2}+\left(\frac{1}{(m+2k)^2}-\frac{1}{(m+2k+1)^2}\right)\right]\\
&= \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\left[\frac{2(-1)^m}{(2m+2k+1)^2}+\left(\frac{1}{(m+2k)^2}-\frac{1}{(m+2k+1)^2}\right)\right]\\
&= \sum_{k=0}^{\infty}\left[\sum_{m=1}^{\infty}\frac{2(-1)^m}{(2m+2k+1)^2}+\sum_{m=1}^{\infty}\left(\frac{1}{(m+2k)^2}-\frac{1}{(m+2k+1)^2}\right)\right]\\
&= \sum_{k=0}^{\infty}\left[\sum_{m=1}^{\infty}\frac{2(-1)^m}{(2m+2k+1)^2}+\frac{1}{(2k+1)^2}\right].\\
&= \sum_{k=0}^{\infty}\left[2\sum_{m=1}^{\infty}(-1)^m\int_0^1\int_0^1(xy)^{2m+2k}\,dydx+\int_0^1\int_0^1(xy)^{2k}\,dydx\right]\\
&= \sum_{k=0}^{\infty}\left[2\int_0^1\int_0^1\left(\sum_{m=1}^{\infty}(-1)^m(xy)^{2m+2k}\right)\,dydx+\int_0^1\int_0^1(xy)^{2k}\,dydx\right].
\end{aligned}$$

The geometric sum in the latter expression can be computed so that $S$ can be written as

$$\begin{aligned} S &= \sum_{k=0}^{\infty}\left[\int_0^1\int_0^1\left(-2\frac{(xy)^{2k+2}}{1+(xy)^2}\right)dydx+\int_0^1\int_0^1 (xy)^{2k}\,dydx\right] \\ &= \sum_{k=0}^{\infty}\left[\int_0^1\int_0^1 (xy)^{2k}\left(1-\frac{2(xy)^2}{1+(xy)^2}\right)dydx\right] \\ &= \int_0^1\int_0^1 \frac{1}{1-(xy)^2}\left(\frac{1-(xy)^2}{1+(xy)^2}\right)dydx = \int_0^1\int_0^1\frac{1}{1+(xy)^2}\,dydx \\ &= \int_0^1\left(\frac{1}{x}\arctan(xy)\right)\Big|_0^1 dx = \int_0^1\frac{\arctan(x)}{x}\,dx \\ &= \int_0^1\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-2}}{2n-1}\,dx \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}. \end{aligned}$$

**Solution 3 by Michel Bataille, Rouen, France.**

Let $a_{m,n} = \dfrac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n+(-1)^{n-m}\left(3+\dfrac{m-3n}{n+m}+\dfrac{n^2}{(m+n)^2}\right)\right)$ and $U_m = \sum_{n=0}^{\infty} a_{m,n}$.
An easy calculation gives

$$a_{m,n} = \frac{(-1)^m(1-(-1)^n)}{(2m+n)^2}+\frac{(-1)^n}{(m+n)^2}$$

so that, the series involved being absolutely convergent,

$$\begin{aligned} U_m &= (-1)^m\sum_{n=0}^{\infty}\frac{(1-(-1)^n)}{(2m+n)^2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(m+n)^2} \\ &= 2(-1)^m\sum_{n=0}^{\infty}\frac{1}{(2m+2n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{(m+2n)^2}-\sum_{n=0}^{\infty}\frac{1}{(m+2n+1)^2}. \end{aligned}$$

For any positive integer $r$, we see that

$$U_{2r-1} = -2\sum_{n=0}^{\infty}\frac{1}{(4r+2n-1)^2}+\sum_{n=0}^{\infty}\frac{1}{(2r+2n-1)^2}-\sum_{n=0}^{\infty}\frac{1}{(2r+2n)^2}$$

$$U_{2r} = 2\sum_{n=0}^{\infty}\frac{1}{(4r+2n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{(2r+2n)^2}-\sum_{n=0}^{\infty}\frac{1}{(2r+2n+1)^2}$$

and therefore

$$U_{2r-1}+U_{2r}=\frac{-2}{(4r-1)^2}+\frac{1}{(2r-1)^2}.$$

Finally

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}a_{m,n} &= \sum_{r=1}^{\infty}\left(\frac{-2}{(4r-1)^2}+\frac{1}{(2r-1)^2}\right)\\ &= \sum_{r=1}^{\infty}\frac{-2}{(4r-1)^2}+\sum_{r=1}^{\infty}\frac{1}{(4r-1)^2}+\sum_{r=0}^{\infty}\frac{1}{(4r+1)^2}\\ &= \sum_{r=0}^{\infty}\frac{1}{(4r+1)^2}-\sum_{r=1}^{\infty}\frac{1}{(4r-1)^2}\\ &= \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{(2r-1)^2}\end{aligned}$$

and we conclude $\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}a_{m,n}=G$.

**Solution 4 by Perfetti Paolo, dipartimento di matematica, Università di “Tor Vergata", Roma, Italy.**

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}(1-(-1)^n)=\sum_{m=1}^{\infty}2(-1)^m\sum_{n=0}^{\infty}\frac{1}{(2m+2n+1)^2}.$$

$$A_n:=\sum_{m=1}^{n}(-1)^m=\frac{(-1)^n-1}{2},\qquad B_m:=\sum_{n=0}^{\infty}\frac{1}{(2m+2n+1)^2}.$$

$$\frac{1}{2m+1}=\int_0^{\infty}\frac{dx}{(2m+2x+1)^2}\leqslant B_m\leqslant\int_{-1}^{\infty}\frac{dx}{(2m+2x+1)^2}=\frac{1}{2m-1}.$$

By Abel’s “Summation by parts" and assuming $n=2N$ even we get

$$\sum_{m=1}^{n}(-1)^m\sum_{n=0}^{\infty}\frac{1}{(2m+2n+1)^2}=\underbrace{A_nB_n}_{\to 0}-\underbrace{A_0B_0}_{=0}+\sum_{m=1}^{n}A_{m-1}(B_{m-1}-B_m)=$$

$$=\frac{1}{2}\sum_{m=1}^{n}(-1)^{m-1}(B_{m-1}-B_m)-\frac{1}{2}\sum_{m=1}^{n}(B_{m-1}-B_m)=$$

$$=\frac{1}{2}\sum_{m=1}^{n}(-1)^{m-1}\frac{1}{(2m-1)^2}-\frac{B_0}{2}+\frac{B_n}{2}\to\frac{G}{2}-\frac{B_0}{2}=\frac{G}{2}-\frac{\pi^2}{16}.$$

Moreover

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}(-1)^{n-m}\left(3+\frac{m-3n}{n+m}+\frac{n^2}{(m+n)^2}\right)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\frac{(2m+n)^2}{(m+n)^2}.$$

Again using Abel's "Summation by parts" and letting $C_m := \sum_{k=0}^{\infty} \frac{1}{(m+k)^2}$, we have

$$\sum_{n=0}^{\infty} \frac{-1}{(n+1)^2} + \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{(-1)^m}{(m+n)^2} =$$
$$= \sum_{n=0}^{\infty} \frac{-1}{(n+1)^2} + A_M C_M - A_1 C_1 + \sum_{m=2}^{M} A_{m-1}(C_{m-1} - C_m) =$$
$$= A_M C_M + \sum_{m=2}^{M} \frac{A_{m-1}}{(m-1)^2} = A_M C_M + \sum_{m=2}^{M} \frac{1-(-1)^{m-1}}{2} \frac{1}{(m-1)^2}$$

whose limit for $M \to \infty$ gives

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

The sum of the two contributions is

$$2\frac{G}{2} + 2\frac{\pi^2}{16} - \frac{\pi^2}{8} = G.$$

**Solution 5 by Péter Fülöp, Gyömrő, Hungary.**

1. Transformations

Start the transformations with $\Big(\underbrace{3 + \frac{m-3n}{n+m}}_{3+1-\frac{4n}{m+n}} + \frac{n^2}{(m+n)^2}\Big)$ term:

$$4 - \frac{4n}{m+n} + \frac{n^2}{(m+n)^2} = (2 - \frac{n}{m+n})^2 = \frac{(2m+n)^2}{(m+n)^2}.$$

We can separate the original sum $S$ into two different sums $S_1, S_2$ in the following way:

$$S_1 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+n)^2}(1-(-1)^n)$$

$$S_2 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+n)^2}(-1)^{n-m} \frac{(2m+n)^2}{(m+n)^2}.$$

Decompose $S_1$ into even and odd parts. $n = 2k$ if n even and $n = 2k+1$ if n odd.

$$S_1 = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^m}{(2m+2k+1)^2} \underbrace{(1-(-1)^{2k+1})}_{2} + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^m}{(2m+2k)^2} \underbrace{(1-(-1)^{2k})}_{0}$$

$$S_1 = 2\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^m}{(2m+2k+1)^2}$$

$$S_2 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\cdot\frac{(-1)^n}{(m+n)^2}$$

2. Integrations

Use the result $\frac{1}{r^2} = \int_0^{\infty} te^{-rt}dt$ and reverse the order of the summation and integration twice:

$$S_1 = 2\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(-1)^m\int_0^{\infty} te^{-t(2m+2k+1)}dt = 2\sum_{m=1}^{\infty}(-1)^m\int_0^{\infty} te^{-t(2m+1)}\underbrace{\sum_{k=0}^{\infty}e^{-2kt}}_{\frac{1}{1-e^{-2t}}}dt,$$

$$S_1 = 2\int_0^{\infty}\underbrace{\sum_{m=1}^{\infty}(-e^{-2t})^m}_{\frac{-e^{-2t}}{1+e^{-2t}}}\frac{te^{-t}}{1-e^{-2t}}dt = -2\int_0^{\infty}\frac{te^{-3t}}{1-e^{-4t}}dt\,.$$

Treating $S_2$ likewise:

$$S_2 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int_0^{\infty} te^{-t(m+n)}dt = \sum_{m=1}^{\infty}\int_0^{\infty} te^{-tm}\underbrace{\sum_{n=0}^{\infty}(-e)^{-nt}}_{\frac{1}{1+e^{-t}}}dt,$$

$$S_2 = \int_0^{\infty}\frac{t}{1+e^{-t}}\underbrace{\sum_{m=1}^{\infty}e^{-tm}}_{\frac{e^{-t}}{1-e^{-t}}}dt = \int_0^{\infty}\frac{te^{-t}}{1-e^{-2t}}dt$$

$$S = S_1 + S_2 = -2\int_0^{\infty}\frac{te^{-3t}}{1-e^{-4t}} + \frac{te^{-t}}{1-e^{-2t}}dt$$

$$S = \int_0^{\infty}\frac{te^{-t}}{1-e^{-2t}}\left(\frac{-2e^{-2t}}{1+e^{-2t}}+1\right)dt = \int_0^{\infty}\frac{t}{1+e^{-2t}}dt.$$

By the substitution: $x = e^{-t}$, we get

$$S = -\int_0^1 \frac{\ln(x)}{1+x^2}dx.$$

Using the results $\ln(x) = \left.\frac{d(x^a)}{da}\right|_{a=0}$ and $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k$, we have

$$-\frac{d}{da}\int_0^1 \sum_{k=0}^{\infty}(-1)^k x^{(2k+a)}dx = -\frac{d}{da}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1+a} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1+a)^2}_{|a=0},$$

$$S = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = G.$$

**Solution 6 by Albert Stadler, Herrliberg, Switzerland.**

We note that

$$3 + \frac{m-3n}{m+n} + \frac{n^2}{(m+n)^2} = \frac{(2m+n)^2}{(m+n)^2}$$

and that

$$\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+2n+1)^2}.$$

Hence

$$S := \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n+(-1)^{m+n}\left(3+\frac{m-3n}{m+n}+\frac{n^2}{(m+n)^2}\right)\right) =$$

$$= 2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+2n+1)^2} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(m+n)^2} =$$

$$= 2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2} - 2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+2n)^2} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(m+n)^2} =$$

$$= 2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(m+n)^2} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(m+n)^2} =$$

$$= 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+n)^2} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m}{(m+n)^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(m+n)^2} + 2\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^2} - \frac{1}{2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2} + \sum_{m=1}^{\infty}\frac{1}{m^2} =$$

$$= 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+n)^2} + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m}{(m+n)^2} + \frac{\pi^2}{6}.$$

We have

$$\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m}{(m+n)^2} = -\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^m\int_0^1 x^{m+n-1}\log x\,dx = -\frac{1}{2}\int_0^1\frac{-x}{1+x}\cdot\frac{1}{1-x}\log x\,dx =$$

$$= \frac{1}{2}\int_0^1\frac{x}{1-x^2}\log x\,dx = \frac{1}{2}\sum_{k=1}^{\infty}\int_0^1 x^{2k-1}\log x\,dx = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{4k^2} = -\frac{\pi^2}{48}$$

and

$$2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+n)^2} = -2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^m\int_0^1 x^{2m+n-1}\log x\,dx = -2\int_0^1\frac{-x^2}{1+x^2}\cdot\frac{1}{1-x}\log x\,dx =$$

$$= \int_0^1\frac{2x^2}{(1+x^2)(1-x)}\log x\,dx = \int_0^1\left(\frac{1}{1-x}-\frac{1+x}{1+x^2}\right)\log x\,dx =$$

$$= \sum_{k=1}^{\infty}\int_0^1 x^{k-1}\log x\,dx - \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1\left(x^{2k-2}+x^{2k-1}\right)\log x\,dx =$$

$$= -\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k)^2} = -\frac{\pi^2}{6}+G+\frac{1}{4}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}-2\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right) =$$

$$= -\frac{\pi^2}{6}+G+\frac{\pi^2}{48}.$$

Thus $S = G$.

**Also solved by and the problem proposer.**

• **5779** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Romania..*

If $0 < a \leqslant b$ then:

$$e^{ab} + e^{(\frac{a+b}{2})^2} \leqslant e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab}+\frac{a+b}{2}-\frac{2ab}{a+b})^2}$$

**Solution 1 by Albert Stadler, Herrliberg, Switzerland.**

We divide both sides by $e^{\left(\frac{2ab}{a+b}\right)^2} > 0$ and get the equivalent inequality

$$e^{\frac{ab(a-b)^2}{(a+b)^2}} + e^{\frac{(a-b)^2\left(a^2+6ab+b^2\right)}{4(a+b)^2}} \leqslant 1 + e^{\frac{(a-b)^2\left(a+4\sqrt{ab}+b\right)}{4(a+b)}}.$$

Let x be real. The Taylor expansion of the exponential function is

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

It is therefore sufficient to prove that

$$\left(\frac{ab(a-b)^2}{(a+b)^2}\right)^k + \left(\frac{(a-b)^2\left(a^2+6ab+b^2\right)}{4(a+b)^2}\right)^k \leqslant \left(\frac{(a-b)^2\left(a+4\sqrt{ab}+b\right)}{4\left(a+b\right)}\right)^k$$

for all k⩾1. This inequality is equivalent to

$$(4ab)^k + \left(a^2+6ab+b^2\right)^k \leqslant \left(\left(a+4\sqrt{ab}+b\right)(a+b)\right)^k,\ \ k=1,\ 2,\ 3,\ \ldots$$

which is true, since

$$\left(a+4\sqrt{ab}+b\right)(a+b) = (a+b)^2+4\sqrt{ab}\,(a+b) \geqslant (a+b)^2+4\sqrt{ab}\cdot 2\sqrt{ab} = a^2+10ab+b^2$$

and given x, y>0 we have $x^k+y^k \leqslant (x+y)^k$ for k=1, 2, 3 ...

**Solution 2 by Michel Bataille, Rouen, France.**

If $a=b$ equality holds, so we suppose that $a<b$ in what follows. Let $h=2ab/(a+b)$, $g=\sqrt{ab}$, $m=(a+b)/2$. We have $h<g<m$ (inequalities of means). The proposed inequality writes as

$$\phi(g)-\phi(h) \leqslant \phi(g+m-h)-\phi(m) \tag{1}$$

where $\phi(x)=e^{x^2}$. Note that $h<g<m<g+m-h$. The Mean Value Theorem shows that $\phi(g)-\phi(h)=(g-h)\phi'(u)$ and $\phi(g+m-h)-\phi(m)=(g-h)\phi'(v)$ for some $u\in(h,g)$, $v\in(m,g+m-h)$. An easy calculation gives $\phi''(x)=(4x^2+2)e^{x^2}>0$, hence $\phi'$ is an increasing function. Consequently $\phi'(u)<\phi'(v)$ and since $g-h>0$, (1) follows.

**Solution 3 by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.**

Set

$$\sqrt{ab} =: G, \quad \frac{a+b}{2} =: M, \quad \frac{2ab}{a+b} =: H.$$

We know $M\geqslant G\geqslant H$. The inequality in question is equivalent to

$$e^{G^2}-e^{H^2} \leqslant e^{(G+M-H)^2}-e^{M^2}.$$

We observe that $(G+M-H)^2-M^2=(G-H)(G+2M-H)\geqslant 0$. Clearly

$$H^2 \leqslant G^2 \leqslant M^2 \leqslant (G+M-H)^2$$

Lagrange's theorem yields

$$e^{G^2}-e^{H^2} = e^{\xi}(G^2-H^2) \leqslant e^{(G+M-H)^2}-e^{M^2} = e^{\eta}\left((G+M-H)^2-M^2\right)$$

where $G^2 < \xi < H^2 \leqslant M^2 < \eta < (G+M-H)^2$. Because of $(e^x)'' = e^x > 0$ we have

$$e^{\eta}\left((G+M-H)^2 - M^2\right) \geqslant e^{\xi}\left((G+M-H)^2 - M^2\right)$$

thus it suffices to show that

$$e^{\xi}(G^2 - H^2) \leqslant e^{\xi}\left((G+M-H)^2 - M^2\right) \iff G^2 - H^2 \leqslant (G+M-H)^2 - M^2$$

or

$$GH + MH \leqslant H^2 + GM \iff H(M-H) \leqslant G(M-H)$$

and this concludes the proof.

**Also solved by Prakash Pant, The University of Vermont, Bardiya, Nepal and the problem proposer.**

• **5780** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $\alpha, \beta, \gamma$ be angles of an arbitrary triangle. Prove that the following inequality holds

$$\frac{\alpha\cos\alpha + \beta\cos\beta + \gamma\cos\gamma}{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma} \leqslant \cot\left(\frac{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma}\right).$$

When does equality occur?

**Solution 1 by Michel Bataille, Rouen, France.**

Let $f$ be the function $f$ defined by $f(x) = x\cot x$. For $0 < x < \pi$, we have

$$f''(x) = \frac{2(x\cos x - \sin x)}{\sin^3 x} < 0$$

since the function $g(x) := x\cos x - \sin x$ on the interval $(0,\pi)$ satisfies $g'(x) = -x\sin x < 0$, hence $g(x) < g(0) = 0$, and therefore $f$ is strictly concave on the interval $(0,\pi)$.

Let $X := \dfrac{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma}$. Since

$$\frac{\sin\alpha}{\sin\alpha + \sin\beta + \sin\gamma} + \frac{\sin\beta}{\sin\alpha + \sin\beta + \sin\gamma} + \frac{\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma} = 1,$$

then Jensen's inequality yields

$$f(X) = X\cot X \geqslant \frac{\sin\alpha}{\sin\alpha + \sin\beta + \sin\gamma}f(\alpha) + \frac{\sin\beta}{\sin\alpha + \sin\beta + \sin\gamma}f(\beta) + \frac{\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma}f(\gamma).$$

Since $(\sin x)f(x) = x\cos x$, the latter can be re-written as

$$X\cot X \geqslant \frac{\alpha\cos\alpha + \beta\cos\beta + \gamma\cos\gamma}{\sin\alpha + \sin\beta + \sin\gamma},$$

or (since $\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma > 0$)

$$\cot X \geqslant \frac{\alpha\cos\alpha + \beta\cos\beta + \gamma\cos\gamma}{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma},$$

as required. Since $f$ is strictly concave, equality holds if and only if $\alpha = \beta = \gamma\,(= \frac{\pi}{3})$.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.**

The function $f(x) := x\cot x$ is concave, for if $-\pi < x < \pi$

$$f''(x) = 2\frac{x}{\sin^2 x}\left(\cot x - \frac{1}{x}\right) = \left(\frac{2x}{\sin x}\right)^2 \sum_{n=1}^{\infty} \frac{1}{x^2 - (n\pi)^2} < 0,$$

where we have used the partial fraction decomposition of the cotangent function

$$\cot x - \frac{1}{x} = \sum_{n=1}^{\infty}\left(\frac{1}{x + n\pi} + \frac{1}{x - n\pi}\right) = 2x\sum_{n=1}^{\infty} \frac{1}{x^2 - (n\pi)^2}.$$

Hence, by Jensen's inequality,

$$\frac{\alpha\cos\alpha + \beta\cos\beta + \gamma\cos\gamma}{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma} = \frac{\sin\alpha\, f(\alpha) + \sin\beta\, f(\beta) + \sin\gamma\, f(\gamma)}{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma} =$$

$$= \frac{\sin\alpha + \sin\beta + \sin\gamma}{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma} \cdot \frac{(\sin\alpha)\, f(\alpha) + (\sin\beta)\, f(\beta) + (\sin\gamma)\, f(\gamma)}{\sin\alpha + \sin\beta + \sin\gamma} \leqslant$$

$$\leqslant \frac{\sin\alpha + \sin\beta + \sin\gamma}{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma} f\left(\frac{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma}\right) =$$

$$= \cot\left(\frac{\alpha\sin\alpha + \beta\sin\beta + \gamma\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma}\right).$$

**Also solved by and the problem proposer.**

***Editor's Statement***: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

*Keep in mind that the examples given below are your best guide!*

## Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

## Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

**#ProblemNumber_FirstName_LastName_Solution_SSMJ**

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

**#1234_Max_Planck_Solution_SSMJ**

**#9876_Charles_Darwin_Solution_SSMJ**

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

**Please adopt the following structure, in the order shown, for the presentation of your solution:**

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

## Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

**FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle**

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

**Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns**

**Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle**

**Please adopt the following structure, in the order shown, for the presentation of your proposal:**

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

*Problem proposed to SSMJ*

*Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*

**Principia Mathematica** (⟵ You may choose to not include a title.)

**Statement of the problem:**

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

**Solution of the problem:** . . . . . .

## ♣ ♣ ♣ Thank You! ♣ ♣ ♣