Problems and Solutions

Albert Natian, Section Editor

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Note: This February 2024 issue replaces the former January 2024 issue.

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please present all proposed solutions and proposed problems according to formatting requirements delineated near the end of this document. Also, please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* May 1, 2024.

• **5763** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina..*

Consider the Diophantine equation in the unknown positive integer n:

$$b_1\left(n+a_1\right)^{dn}+b_2\left(n+a_2\right)^{dn}+\cdots+b_k\left(n+a_k\right)^{dn}=c$$

where k, d, c and a_i, b_i $(i=1,2,\ldots,k)$ are given fixed integers with $k\geqslant 2$, $d>0$, $b_i\neq 0$ and $a_1<a_2<\cdots<a_k$. Prove that the solution set (possibly empty) of the above Diophantine equation is finite.

• **5764** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Let F_n denote the nth term of the Fibonacci sequence defined by $F_0=0$, $F_1=1$, $F_{n+2}=F_{n+1}+F_n$ for $n\geqslant 0$. Let $(a_n)_{n\geqslant 1}$ be any sequence of positive real numbers with $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2a_n}=2\pi$. Evaluate :

$$\Lambda=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(2n+1)!!}}-\sqrt[n]{\frac{a_nF_n}{(2n-1)!!}}\right).$$

• **5765** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Let $s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim_{n\to\infty} s_n = s$ (Ioachimescu Constant). For non-negative intger m, evaluate:

$$L = \lim_{n\to\infty}\left(s^{m+1} - \prod_{j=n}^{n+m} s_j\right).$$

• **5766** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Show that

$$\left(\frac{2}{\pi}\int_0^{\pi}\frac{\sin^2 x}{x^2}\text{dx}\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi}\frac{\sin^2 x}{x^2}\text{dx}\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi}\frac{\sin^2 x}{x^2}\text{dx}\right) < \left(\frac{1}{3}\right)^3.$$

• **5767** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Let a, b, c, d be nonnegative real numbers such that

$$ab + bc + cd + da = 4, \qquad a \geqslant b \geqslant c \geqslant d.$$

Prove that

$$\frac{1}{ab+7} + \frac{1}{ac+7} + \frac{1}{ad+7} + \frac{1}{bc+7} + \frac{1}{bd+7} + \frac{1}{cd+7} \geqslant \frac{3}{4}.$$

• **5768** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$I = \int_0^{\infty}\frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx.$$

Solutions

To Formerly Published Problems

• **5739** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Prove that for any triangle $\triangle ABC$:

$$\frac{h_a^3}{h_b^3} + \frac{h_b^3}{h_c^3} + \frac{h_c^2}{h_a^3} \geqslant \frac{\sin^2 B}{\sin^2 A} + \frac{\sin^2 C}{\sin^2 B} + \frac{\sin^2 A}{\sin^2 C}$$

where h_a, h_b, h_c are the altitudes respectively issued from the vertices A, B, C.

Solution 1 by Michel Bataille, Rouen, France.

Let F and R be the area and the circumradius of the triangle. Let $a = BC, b = CA, c = AB$. Since $ah_a = bh_b = ch_c = 2F$ and $2R\sin A = a, 2R\sin B = b, 2R\sin C = c$, the inequality is equivalent to

$$\frac{b^3}{a^3} + \frac{c^3}{b^3} + \frac{a^3}{c^3} \geqslant \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}. \tag{1}$$

From an inequality of means, we have

$$\frac{b^3}{a^3} + \frac{c^3}{b^3} + \frac{a^3}{c^3} \geqslant \frac{1}{\sqrt{3}}\left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right)^{3/2} \tag{2}$$

and from AM-GM, we have

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \geqslant 3\left(\frac{b^2}{a^2} \cdot \frac{c^2}{b^2} \cdot \frac{a^2}{c^2}\right)^{1/3} = 3$$

hence,

$$\left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right)^{3/2} = \left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right)^{1/2}\left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right) \geqslant \sqrt{3} \cdot \left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right).$$

Combining with (2), the desired inequality (1) follows.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By the law of sines,

$$\frac{h_a}{h_b} = \frac{b}{a} = \frac{\sin B}{\sin A}, \quad \frac{h_b}{h_c} = \frac{c}{b} = \frac{\sin C}{\sin B}, \quad \frac{h_c}{h_a} = \frac{a}{c} = \frac{\sin A}{\sin C}.$$

So

$$\frac{h_a^3}{h_b^3} + \frac{h_b^3}{h_c^3} + \frac{h_c^3}{h_a^3} = \frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}.$$

By Hölder's inequality,

$$\frac{\sin^2 B}{\sin^2 A} + \frac{\sin^2 C}{\sin^2 B} + \frac{\sin^2 A}{\sin^2 C} \leqslant \left(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\right)^{\frac{2}{3}}(1+1+1)^{\frac{1}{3}}.$$

It remains to prove that

$$\left(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\right)^{\frac{2}{3}}(1+1+1)^{\frac{1}{3}} \leqslant \frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}$$

which is equivalent to $\dfrac{\sin^3 B}{\sin^3 A} + \dfrac{\sin^3 C}{\sin^3 B} + \dfrac{\sin^3 A}{\sin^3 C} \geqslant 3$. However this inequality follows from the AM-GM inequality:

$$\frac{1}{3}\left(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\right) \geqslant \frac{\sin B}{\sin A} \cdot \frac{\sin C}{\sin B} \cdot \frac{\sin A}{\sin C} = 1.$$

Also solved by the problem proposer.

• **5740** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

How many integers m are there for which the volume of parallelepiped determined by vectors $u = \langle 2023,\, m,\, 1-m\rangle$, $v = \langle m,\, 2-m,\, 4046\rangle$, $w = \langle 6069,\, 3-m,\, m\rangle$ is equal to $6(2023^2 + 2023)$?

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

There is only one such integer: $m = 0$.

The volume of the parallelepiped is given by $\left|u \cdot (v \times w)\right|$, which is the absolute value of the determinant of the matrix

$$M = \begin{pmatrix} 2023 & m & 1-m \\ m & 2-m & 4046 \\ 6069 & 3-m & m \end{pmatrix}.$$

The determinant of M is given by

$$\det(M) = -4 \cdot 2024m^2 + \left(8 \cdot 2023^2 + 11 \cdot 2023 + 3\right) m - 6\left(2023^2 + 2023\right).$$

If $\det(M) < 0$, then we require

$$\begin{aligned} 4 \cdot 2024m^2 - \left(8 \cdot 2023^2 + 11 \cdot 2023 + 3\right) m + 6\left(2023^2 + 2023\right) &= 6\left(2023^2 + 2023\right) \\ m\left(4 \cdot 2024m - \left(8 \cdot 2023^2 + 11 \cdot 2023 + 3\right)\right) &= 0 \\ 4m\left(2024m - \left(2 \cdot 2023^2 + 5564\right)\right) &= 0, \end{aligned}$$

so that either $m = 0$ or $m = \dfrac{16187}{4} \notin \mathbb{Z}$.

On the other hand, if $\det(M) \geqslant 0$, then we must have

$$\begin{aligned} -4 \cdot 2024m^2 + \left(8 \cdot 2023^2 + 11 \cdot 2023 + 3\right) m - 6\left(2023^2 + 2023\right) &= 6\left(2023^2 + 2023\right) \\ 2024\left(4m^2 - 16187m + 24276\right) &= 0. \end{aligned}$$

The discriminant of this quadratic equation is

$$16,187^2 - 16 \cdot 24,276 = 261,630,553,$$

which is not a perfect square, so the solutions to this equation are not integers. Therefore, the only integer m that satisfies the volume condition is $m = 0$.

Solution 2 by Michel Bataille, Rouen, France.

We show that there is only one possibility for m, namely $m = 0$.

We want the volume $6(2023^2 + 2023)$ to be equal to $|\delta|$ where δ is the determinant

$$\begin{vmatrix} a & m & 3a \\ m & 2-m & 3-m \\ 1-m & 2a & m \end{vmatrix}$$

(setting $a = 2023$).
Adding the rows 2 and 3 to the first one and other manipulations give

$$\delta = (a+1)\begin{vmatrix} 1 & 2 & 3 \\ m & 2-m & 3-m \\ 1-m & 2a & m \end{vmatrix} = (a+1)\begin{vmatrix} 1-m & m & m \\ m & 2-m & 3-m \\ 1-m & 2a & m \end{vmatrix} = (a+1)\begin{vmatrix} 0 & m-2a & 0 \\ m & 2-m & 3-m \\ 1-m & 2a & m \end{vmatrix},$$

that is, $\delta = (a+1)(m-2a)(3-4m)$.
Thus, we are looking for integers m such that $2024(m - 4046)(3 - 4m) = 6 \cdot 2023 \cdot 2024$ or $2024(m - 4046)(3 - 4m) = -6 \cdot 2023 \cdot 2024$.
The latter leads to the equation $m(4m - 8a - 3) = 0$ whose only integral solution is $m = 0$; the former leads to $4m^2 - (8a+3)m + 12a = 0$. But $\Delta = (8a+3)^2 - 4 \cdot 4 \cdot 12a = 64(2023)^2 - 144 \cdot 2023 + 9$ is not a perfect square (since $\Delta \equiv 3 \pmod{10}$), hence $4m^2 - (8a + 3)m + 12a = 0$ has no integral solution for m. The announced result follows.

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.

The volume of the parallelpiped is equal to $(u \times v) \cdot w$. Now $u \times v$ is

$$\begin{vmatrix} i & j & k \\ 2023 & m & 1-m \\ m & 2-m & 4046 \end{vmatrix} = \left[4046m - (1-m)(2-m)\right] i - \left[((2023)(4046) - m(1-m)\right] j + \left[2023(2-m) - m^2\right] k.$$

So $(u \times v) \cdot w$ is

$$6069\left[4046m-(1-m)(2-m)\right]-(3-m)\left[((2023)(4046)-m(1-m)\right]$$
$$+m\left[2023(2-m)-m^2\right].$$

Expanding and simplifying, this becomes

$$2024(4046-m)(4m-3).$$

Since $6(2023^2+2023)=6(2023)(2023+1)=6(2023)(2024)$, we seek integers m such that

$$(4046-m)(4m-3)=6(2023).$$

This quadratic equation, which can be rewritten $4m^2-16,187m+24,276=0$, has the two solutions $m=\dfrac{16,187\pm\sqrt{261,630,553}}{8}$, neither of which is an integer. So there is no integer m that fits the criteria.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

The volume of the parallelepiped spanned by the vectors u, v, and w is given by $|u\cdot(v\times w)|$. Now,

$$\begin{aligned} u\cdot(v\times w) &= \begin{vmatrix} 2023 & m & 1-m \\ m & 2-m & 4046 \\ 6069 & 3-m & m \end{vmatrix} \\ &= -(2023+1)(4m^2-(8\cdot 2023+3)m+6\cdot 2023). \end{aligned}$$

If $u\cdot(v\times w)=-6(2023^2+2023)$, then $4m^2-(8\cdot 2023+3)m=0$ and

$$m=0 \quad \text{or} \quad m=\frac{16187}{4}.$$

On the other hand, if $u\cdot(v\times w)=6(2023^2+2023)$, then $4m^2-(8\cdot 2023+3)m+12\cdot 2023=0$ and

$$m=\frac{16187\pm\sqrt{261630533}}{8}.$$

Thus, there is only one integer m for which the volume of the parallelepiped determined by the vectors $u=\langle 2023,m,1-m\rangle$, $v=\langle m,2-m,4046\rangle$, $w=\langle 6069,3-m,m\rangle$ is equal to $6(2023^2+2023)$, and that is $m=0$.

Solution 5 by Brian D. Beasley, Simpsonville, SC.

We show that the only such integer is $m = 0$.

Let $n = 2023$. Then the volume of the parallepiped determined by the three given vectors is

$$V = \left| \det \begin{bmatrix} n & m & 3n \\ m & 2-m & 3-m \\ 1-m & 2n & m \end{bmatrix} \right|.$$

Thus in order to obtain $V = 6(n^2 + n)$, we require

$$|(n+1)(8mn + 3m - 4m^2 - 6n)| = 6n(n+1).$$

Since $n \neq -1$, we consider two cases:

(i) If $8mn + 3m - 4m^2 - 6n = 6n$, then

$$n = \frac{4m^2 - 3m}{8m - 12}.$$

But $n = 2023$ produces non-integral values for m.

(ii) If $8mn + 3m - 4m^2 - 6n = -6n$, then

$$m(8n + 3 - 4m) = 0.$$

For $n = 2023$, $8n + 3 - 4m$ cannot be zero for any integer m. Hence we conclude $m = 0$.

Addendum. To generalize, we let $u = \langle n, m, 1-m \rangle$, $v = \langle m, 2-m, 2n \rangle$, and $w = \langle 3n, 3-m, m \rangle$, where n is an arbitrary integer, and we seek all integers m that produce a parallelepiped with volume $V = 6(n^2 + n)$.

If $n = -1$, then

$$(m^2 - 6)u + (4m - 3)v = (-m^2 + m - 2)w,$$

so the three vectors are not linearly independent.

Next, for case (i) above, we show that if m and n are integers, then $(m, n) = (0, 0)$: If $8m - 12$ divides $4m^2 - 3m$, then it also divides

$$36 = 8(4m^2 - 3m) - (8m - 12)(4m + 3).$$

Since $8m - 12$ divides 36 and is divisible by 4, we have

$$8m - 12 \in \{\pm 4, \pm 12, \pm 36\} \quad \Longrightarrow \quad m \in \{-3, 0, 1, 2, 3, 6\}.$$

But 4 must also divide m, so we conclude that $m = 0$ and thus $n = 0$. Hence in this case, $3v = 2w$, so the three vectors are not linearly independent.

Finally, we observe that for case (ii) above, $8n + 3 - 4m \neq 0$ for all integers m and n, as otherwise $3 = 4(m - 2n)$. Thus to produce an actual parallelepiped with the desired volume, we must let n be an integer with $n \notin \{-1, 0\}$ and take $m = 0$.

Solution 6 by Albert Stadler, Herrliberg, Switzerland.

The volume of the parallelepiped equals

$$\left| det \begin{pmatrix} 2023 & m & 1-m \\ m & 2-m & 4046 \\ 6069 & 3-m & m \end{pmatrix} \right| = \left| -8096m^2 + 32762488m - 24567312 \right|.$$

The equation $-8096m^2 + 32762488m - 24567312 = 6\left(2023^2 + 2023\right)$ has no integer roots.

The equation $-8096m^2 + 32762488m - 24567312 = -6\left(2023^2 + 2023\right)$ has the rational roots m=0 and m=16187/4.

So the only integer m for which the volume of the given parallelepiped is equal to $6(2023^2+2023)$ is m=0.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; and the problem proposer.

• **5741** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Let $(a_k)_{k=1}^{\infty}$ be a non-decreasing sequence with $0 < a_k \leqslant k$. Define $A_n := \sum_{k=1}^{n} a_k$ and $S_n := \sum_{k=1}^{n} a_k^{\alpha}$ for $\alpha > 1$. Determine whether or not the following series converges:

$$\sum_{k=1}^{\infty} S_k \left(\frac{1}{A_k^{\alpha}} - \frac{1}{A_{k+1}^{\alpha}} \right).$$

Solution by the problem proposer.

Proof Let's assume that the series

$$\sum_{k=1}^{\infty} \left(\frac{a_k}{A_k} \right)^{\alpha} \tag{1}$$

converges. We can prove

$$\lim_{n\to\infty} \frac{S_n}{A_n^{\alpha}} = 0 \tag{2}$$

Proof of (2)

$$
\begin{aligned}
\frac{S_n}{A_n^\alpha} &= \sum_{k=1}^{n} \frac{a_k^\alpha}{\left(\sum_{j=1}^{n/2} a_j + \sum_{j=n/2+1}^{n} a_j\right)^\alpha} = \\
&= \sum_{k=1}^{n/2} \frac{a_k^\alpha}{\left(\sum_{j=1}^{n/2} a_j + \sum_{j=n/2+1}^{n} a_j\right)^\alpha} + \sum_{k=\frac{n}{2}}^{n} \frac{a_k^\alpha}{\left(\sum_{j=1}^{n/2} a_j + \sum_{j=n/2+1}^{n} a_j\right)^\alpha} \leqslant \\
&\leqslant \sum_{k=1}^{n/2} \frac{a_k^\alpha}{\left(\sum_{j=n/2+1}^{n} a_j\right)^\alpha} + \sum_{k=\frac{n}{2}}^{n} \frac{a_k^\alpha}{\left(\sum_{j=1}^{n/2} a_j\right)^\alpha} \leqslant \frac{n}{2} \frac{a_{n/2}}{\left(a_{1+n/2}(n/2)\right)^\alpha} + \sum_{k=\frac{n}{2}}^{n} \frac{a_k^\alpha}{\left(\sum_{j=1}^{k} a_j\right)^\alpha} \leqslant \\
&\leqslant \left(\frac{n}{2}\right)^{1-\alpha} + \sum_{k=\frac{n}{2}}^{n} \frac{a_k^\alpha}{\left(\sum_{j=1}^{k} a_j\right)^\alpha} \leqslant \varepsilon + \varepsilon \qquad \text{q.e.d.}
\end{aligned}
$$

Now we employ Abel's summation by parts

$$
\sum_{k=m}^{n} \frac{a_k^\alpha}{A_k^\alpha} = \frac{S_n}{A_n^\alpha} - \frac{S_m}{A_m^\alpha} - \sum_{k=m}^{n} S_{k-1}\left(\frac{1}{A_{k-1}^\alpha} - \frac{1}{A_k^\alpha}\right)
$$

whence

$$
\sum_{k=m}^{n} S_{k-1}\left(\frac{1}{A_{k-1}^\alpha} - \frac{1}{A_k^\alpha}\right) = \frac{S_n}{A_n^\alpha} - \frac{S_m}{A_m^\alpha} - \sum_{k=m}^{n} \frac{a_k^\alpha}{A_k^\alpha}
$$

Now by (2) for any m (and then n) large enough

$$
0 < \frac{S_n}{A_n^\alpha} < \varepsilon, \quad 0 < \frac{S_m}{A_m^\alpha} < \varepsilon
$$

and by (1)

$$
0 < \sum_{k=m}^{n} \frac{a_k^\alpha}{A_k^\alpha} < \varepsilon \quad \text{(Cauchy criterion)}
$$

hence (2) together with the convergence of (1) yield the result.

The last step is the convergence in (1) which is the content of D.Borwein, American Mathematical Monthly, Vol.72, No.6.(Jun.–Jul., 1965), pp.675–677. The result is

Proposition *Let* $\{a_k\}_{k\geqslant 1}$ *be a sequence such that* $0 \leqslant a_k \leqslant a_{k+1} \leqslant k+1$. *Then for* $\alpha > 1$ *the series* $\sum_{k=1}^{\infty} \left(\frac{a_k}{A_k}\right)^\alpha$ *converges.*

Borwein's proof If $a_k \to l$, then $a_k \leqslant Cl$ and $A_k \geqslant Ckl$ for a suitable positive C whence

$$
\sum_{k=1}^{\infty} \left(\frac{a_k}{A_k}\right)^\alpha \leqslant \sum_{k=1}^{\infty} \left(\frac{Cl}{Clk}\right)^\alpha < +\infty
$$

Let $a_k \to +\infty$ and let N_1 be the set of integers such that $S_k \geqslant ka_k/2$, $k \in N_1$.

$$\sum_{k \in N_1}^{\infty} \left(\frac{a_k}{A_k}\right)^{\alpha} \leqslant \sum_{k \in N_1}^{\infty} \left(\frac{2a_k}{ka_k}\right)^{\alpha} < +\infty$$

Let N_2 be the set of integers such that $S_k < ka_k/2$. We need a Lemma

Lemma *Let* $d_k \geqslant 0$, $d_0 = 0$, $D_n = \sum_{k=1}^{n} d_k \to +\infty$. *Let* $f : [x_0, +\infty)$ *be a decreasing function such that* $\int_{x_0}^{+\infty} f(x)dx < +\infty$. *Then* $\sum_{k=1}^{+\infty} d_k f(D_k)$ *converges.*

Proof of the Lemma

$$\sum_{k=1}^{+\infty} d_k f(D_k) = \sum_{k=1}^{+\infty}(D_k - D_{k-1}) f(D_k) \leqslant \sum_{k=1}^{+\infty} \int_{D_{k-1}}^{D_k} f(x)dx = \int_{d_0}^{+\infty} f(x)dx.$$

End of the proof of the Lemma

Let $d_1 = a_1$, $d_n = \dfrac{A_n}{n} - \dfrac{A_{n-1}}{n-1} = \dfrac{na_n - A_n}{n(n-1)} > 0$. $D_n = \sum_{k=1}^{n} d_k = \dfrac{A_n}{n} \to +\infty$. Indeed

$$\frac{S_n}{n} \geqslant \frac{a_{\left[\frac{n}{2}\right]+1} + \ldots + a_n}{n} \geqslant \frac{\left(n - \left[\frac{n}{2}\right] - 1\right)}{n} a_{1+\left[\frac{n}{2}\right]} \geqslant \frac{1}{3} a_{1+\left[\frac{n}{2}\right]} \to +\infty$$

Moreover

$$\frac{S_k}{k} - \frac{A_{k-1}}{k-1} \geqslant \frac{A_k}{k}\left(\frac{a_k}{A_k} - \frac{1}{k}\right)$$

Indeed it is equivalent to

$$\frac{A_{k-1}}{k-1} \leqslant \frac{A_{k-1}}{k} + \frac{A_k}{k^2} \iff \frac{A_{k-1}}{k(k-1)} \leqslant \frac{A_k}{k^2} \iff A_k \leqslant ka_k$$

and this evidently holds true.

It follows by the Lemma that $\sum_{k \in N_2} \dfrac{d_k}{D_k^\alpha}$ converges since $\int_1^{+\infty} \dfrac{dx}{x^\alpha}$ converges for $\alpha > 1$. Moreover

$$+\infty > \sum_{k \in N_2} \frac{d_k}{D_k^\alpha} = \sum_{k \in N_2} \left(\frac{A_k}{k} - \frac{A_{k-1}}{k-1}\right) \frac{k^p}{A_k^\alpha} \geqslant \sum_{k \in N_2} \frac{A_k}{k}\left(\frac{a_k}{S_k} - \frac{1}{k}\right) \frac{k^\alpha}{A_k^\alpha} \geqslant$$
$$\geqslant \sum_{k \in N_2} \frac{A_k}{a_k}\left(\frac{a_k}{A_k} - \frac{1}{k}\right) \frac{a_k^\alpha}{A_k^\alpha} = \sum_{k \in N_2}\left(1 - \frac{A_k}{ka_k}\right) \frac{a_k^\alpha}{A_k^\alpha} \geqslant \sum_{k \in N_2} \frac{1}{2} \frac{a_k^\alpha}{A_k^\alpha}$$

There was no solution submitted by any other problem solver.

• **5742** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, and Romania, Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Suppose $(x_n)_{n \geqslant 1}$ is a sequence of postive terms with $\lim_{n\to\infty} x_n/n! = \pi$. Define $E_n := \sum_{k=0}^{n} 1/k!$ and let $0 < a \neq 1$. If $f : (0, \infty) \to (0, \infty)$ is a continuous function, then compute

$$L = \lim_{n\to\infty} x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\, dx.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Let F be an anti-derivative of f and define $g(x) = a^x$. By the mean value theorem, there is a real number ξ_n with $E_n < \xi_n < E_{n+1}$ such that

$$\begin{aligned} x_{n+1}\int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx &= x_{n+1}\cdot\Big(F\big(g(E_{n+1})\big) - F\big(g(E_n)\big)\Big) \\ &= x_{n+1}(E_{n+1}-E_n)\cdot(F\circ g)'(\xi_n) \\ &= \frac{x_{n+1}}{(n+1)!}\cdot F'\big(g(\xi_n)\big)\cdot g'(\xi_n). \end{aligned}$$

Noting that $\lim_{n\to\infty} E_n = e$ we obtain

$$L = \pi\cdot f\big(g(e)\big)\cdot g'(e) = \pi\cdot f(a^e)\cdot a^e \ln a.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By the mean value theorem for integrals there is a value $\in \left[a^{E_n}, a^{E_{n+1}}\right]$ such that

$$\int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx = f()\left(a^{E_{n+1}} - a^{E_n}\right).$$

By the mean value theorem there is a value $\in [E_n, E_{n+1}]$ such that

$$a^{E_{n+1}} - a^{E_n} = (\log a)\, a\,(E_{n+1}-E_n) = (\log a)\, a\frac{1}{(n+1)!}.$$

So

$$L = \lim_{n\to\infty} x_{n+1}\int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx = \lim_{n\to\infty} x_{n+1} f()\,(\log a)\, a\frac{1}{(n+1)!} = \pi f(a^e)(\log a)\, a^e.$$

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Because f is continuous on the closed interval $[a^{E_n}, a^{E_{n+1}}]$, by the Extreme Value Theorem, there exist numbers $c_m, c_M \in [a^{E_n}, a^{E_{n+1}}]$ such that

$$f(c_m) = \min_{x\in[a^{E_n},a^{E_{n+1}}]} f(x) \quad\text{and}\quad f(c_M) = \max_{x\in[a^{E_n},a^{E_{n+1}}]} f(x).$$

It then follows that

$$x_{n+1}f(c_m)\left(a^{E_{n+1}} - a^{E_n}\right) \leqslant x_{n+1}\int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx \leqslant x_{n+1}f(c_M)\left(a^{E_{n+1}} - a^{E_n}\right).$$

Now,

$$\lim_{n\to\infty} E_n = e,$$

and, for large n,

$$a^{E_{n+1}} - a^{E_n} = a^{E_n}\left(a^{1/(n+1)!} - 1\right) \sim a^{E_n}\left(\frac{\ln a}{(n+1)!} + O\left(\frac{1}{((n+1)!)^2}\right)\right),$$

so

$$\lim_{n\to\infty} f(c_m) = \lim_{n\to\infty} f(c_M) = f(a^e)$$

and

$$\lim_{n\to\infty} x_{n+1}\left(a^{E_{n+1}} - a^{E_n}\right) = \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)!} \cdot a^{E_n} \ln a = \pi a^e \ln a.$$

Finally, by the squeeze theorem,

$$L = \lim_{n\to\infty} x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx = \pi a^e f(a^e) \ln a.$$

Solution 4 by Michel Bataille, Rouen, France.

We claim that $L = (\pi \ln a) \cdot a^e f(a^e)$.

The change of variables $x = a^t$ gives

$$\int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx = (\ln a) \int_{E_n}^{E_{n+1}} a^t f(a^t)\,dt.$$

Let G denote a primitive of the continuous function $g(t) = a^t f(a^t)$ on $(0, \infty)$. Then we have

$$\int_{E_n}^{E_{n+1}} a^t f(a^t)\,dt = G(E_{n+1}) - G(E_n) = (E_{n+1} - E_n)G'(\theta_n) = \frac{1}{(n+1)!} \cdot a^{\theta_n} f(a^{\theta_n})$$

where $\theta_n \in (E_n, E_{n+1})$.
If $a > 1$, then $a^{E_n} \leqslant a^{\theta_n} \leqslant a^{E_{n+1}}$, while $a^{E_{n+1}} \leqslant a^{\theta_n} \leqslant a^{E_n}$ if $a < 1$.
Since $\lim_{n\to\infty} E_n = \lim_{n\to\infty} E_{n+1} = e$, we see that $\lim_{n\to\infty} a^{\theta_n} = a^e$ (in any case) and since f is continuous, $\lim_{n\to\infty} a^{\theta_n} f(a^{\theta_n}) = a^e f(a^e) > 0$. We deduce that

$$\int_{E_n}^{E_{n+1}} a^t f(a^t)\,dt \ \sim\ \frac{a^e f(a^e)}{(n+1)!} \quad \text{as } n \to \infty.$$

Since $x_{n+1} \ \sim\ (n+1)!\pi$, we obtain

$$x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx \ \sim\ (n+1)!\pi \cdot (\ln a) \frac{a^e f(a^e)}{(n+1)!} = (\pi \ln a) \cdot a^e f(a^e)$$

as $n \to \infty$ and the claim follows.

Also solved by the problem proposer.

• **5743** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Prove that

$$\int_1^{\infty}\left(\frac{\sum_{k=1}^{\infty}\{kx\}\frac{H_k}{k^8}}{x^9}\right)dx = \frac{\zeta(7)}{8} + \frac{3}{28}\zeta(3)\zeta(5) - \frac{\pi^8}{39200}$$

where $\{.\}$ denotes the Fractional Part, H_k denotes the Harmonic Number and $\zeta(s)$ denotes the Riemann Zeta Function.

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $y = \frac{1}{x}, \quad x = \frac{1}{y}$,

$$I = \int_1^{\infty}\left(\frac{\sum\limits_{k=1}^{\infty}\{kx\}\frac{H_k}{k^8}}{x^9}\right)\mathrm{d}x = \int_0^1\left(\frac{\sum\limits_{k=1}^{\infty}\{\frac{k}{y}\}\frac{H_k}{k^8}}{\frac{1}{y^9}}\right)(\frac{1}{y^2})\mathrm{d}y = \int_0^1\left(\sum_{k=1}^{\infty}\frac{H_k}{k^8}\{\frac{k}{y}\}y^7\right)\mathrm{d}y$$

Now evaluate $J = \int_0^1\{\frac{k}{x}\}x^7\mathrm{d}x$. Let $\frac{k}{x} = t$.

$$\begin{aligned}
J &= \int_k^{\infty}\{t\}\frac{k^7}{t^7}\times\frac{k}{t^2}\mathrm{d}t = \int_k^{\infty}\frac{k^8}{t^9}\{t\}\mathrm{d}t = k^8\left\{\sum_{l=k}^{\infty}\int_l^{l+1}\frac{t-1}{t^9}\mathrm{d}t\right\}\\
&= k^8\sum_{l=k}^{\infty}\left\{\int_l^{l+1}\frac{1}{t^8}\mathrm{d}t - l\int_l^{l+1}\frac{1}{t^9}\mathrm{d}t\right\}\\
&= k^8\sum_{l=k}^{\infty}\left\{\frac{1}{7}\left[\frac{1}{l^7}-\frac{1}{(l+1)^7}\right]-\frac{l}{8}\left[\frac{1}{l^8}-\frac{1}{(l+1)^8}\right]\right\}\\
&= k^8\sum_{l=k}^{\infty}\left\{\frac{1}{56l^7}-\frac{1}{56(l+1)^7}-\frac{1}{8(l+1)^8}\right\}\\
&= k^8\left\{\frac{1}{56k^7}-\frac{1}{8}\sum_{l=k}^{\infty}\frac{1}{(l+1)^8}\right\}\\
&= \frac{k}{56}-\frac{k^8}{8}\left[\zeta(8)-\sum_{l=1}^{k}\frac{1}{l^8}\right]\\
\therefore\quad I &= \sum_{k=1}^{\infty}\frac{H_k}{k^8}\left\{\frac{k}{56}-\frac{k^8}{8}\left[\zeta(8)-\sum_{l=1}^{k}\frac{1}{l^8}\right]\right\}
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k^7 \cdot 56} - \frac{1}{8} \sum_{k=1}^{\infty} H_k \left[\zeta(8) - \sum_{l=1}^{k} \frac{1}{l^8} \right]$$

$$= \frac{1}{56} \left[\frac{\pi^8}{4200} - \zeta(3)\zeta(5) \right] - \frac{1}{8} S \text{ in which } S = \sum_{k=1}^{\infty} \left[\zeta(8) - \sum_{l=1}^{k} \frac{1}{l^8} \right] H_k$$

Now evaluate $S = \sum_{k=1}^{\infty} \left[\zeta(8) - \sum_{l=1}^{k} \frac{1}{l^8} \right] H_k$. Let $a_n = H_n, \quad b_n = \sum_{l=n+1}^{\infty} \frac{1}{l^8}$

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} H_k = (n+1)(H_{n+1} - 1), \quad A_k = (k+1)(H_{k+1} - 1)$$

According to Abel Theorem $\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1})$

$$S = \lim_{n\to\infty} \left[(n+1)(H_{n+1} - 1) \sum_{l=n+2}^{\infty} \frac{1}{l^8} \right] + \sum_{k=1}^{\infty} \left[\frac{1}{(k+1)^8} \right] (k+1)(H_{k+1} - 1)$$

$$= \sum_{k=1}^{\infty} \left[\frac{H_{k+1}}{(k+1)^7} - \frac{1}{(k+1)^7} \right]$$

$$= \sum_{k=1}^{\infty} \left[\frac{H_k + \frac{1}{k+1}}{(k+1)^7} - \frac{1}{(k+1)^7} \right]$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^7} + \sum_{k=1}^{\infty} \frac{1}{(k+1)^8} - \sum_{k=1}^{\infty} \frac{1}{(k+1)^7}$$

$$= \frac{\pi^8}{7560} - \zeta(3)\zeta(5) + \zeta(8) - 1 - (\zeta(7) - 1)$$

$$= \frac{\pi^8}{7560} - \zeta(3)\zeta(5) + \zeta(8) - \zeta(7)$$

$$I = \frac{\pi^8}{56 \times 4200} - \frac{1}{56}\zeta(3)\zeta(5) - \frac{\pi^8}{8 \times 7560} + \frac{1}{8}\zeta(3)\zeta(5) - \frac{1}{8}\zeta(8) + \frac{1}{8}\zeta(7)$$

$$= \frac{\zeta(7)}{8} + \frac{3}{28}\zeta(3)\zeta(5) + \pi^8 \left(\frac{1}{56 \times 4200} - \frac{1}{8 \times 7560} - \frac{1}{8 \times 9450} \right)$$

$$= \frac{\zeta(7)}{8} + \frac{3}{28}\zeta(3)\zeta(5) - \frac{\pi^8}{39200}.$$

Q.E.D.

Solution 2 by Narendra Bhandari, Bajura, Nepal.

Interchange of integral and sum yields

$$A = \int_0^{\infty} \left(\frac{1}{x^9} \sum_{k=1}^{\infty} \{kx\} \frac{H_k}{k^8} \right) dx = \sum_{k=1}^{\infty} \frac{H_k}{k^8} \int_1^{\infty} \frac{\{kx\}}{x^9} \, dx.$$

Let $\mathcal{I}$ be the integral. Then enforcing the substitution $kx \to x$ yields

$$\mathcal{I} = k^8 \int_k^\infty \frac{\{x\}}{x^9}\,dx = k^8 \sum_{j=k}^\infty \int_j^{j+1} \frac{x-j}{x}\,dx = k^8 \sum_{j=k}^\infty \left[\frac{j}{8x^8} - \frac{1}{7x^7}\right]_j^{j+1}$$

$$= k^8 \sum_{j=k}^\infty \left(\frac{j}{8(j+1)^8} - \frac{1}{8j^7} - \frac{1}{7}\left(\frac{1}{(j+1)^7} - \frac{1}{j^7}\right)\right) = k^8 \sum_{j=k}^\infty \left(\frac{j}{8(j+1)^8} - \frac{1}{7(j+1)^7}\right)$$

$$+ k^8 \sum_{j=k}^\infty \left(\frac{1}{7j^7} + \frac{1}{8j^7}\right) = k^8 \sum_{j=k}^\infty \left(-\frac{1}{56(j+1)^7} - \frac{1}{8(j+1)^8} + \frac{1}{56j^7}\right) = \frac{k^8}{56} \underbrace{\sum_{j=k}^\infty \left(\frac{1}{j^7} - \frac{1}{(j+1)^7}\right)}_{\text{telescoping series}}$$

$$- \frac{k^8}{8} \sum_{j=k}^\infty \frac{1}{(j+1)^8} = \frac{k^8}{56k^7} - \frac{1}{8}\left(\zeta(8) - \sum_{j=0}^{k-1} \frac{1}{(j+1)^8}\right) = \frac{k}{56} - \frac{1}{8}\left(\zeta(8) - \sum_{j=0}^{k-1} \frac{1}{(j+1)^8}\right).$$

Therefore,

$$A = \frac{1}{56}\sum_{k=1}^\infty \frac{H_k}{k^7} - \frac{1}{8}\sum_{k=1}^\infty H_k \left(\zeta(8) - \sum_{j=0}^{k-1} \frac{1}{(j+1)^8}\right) = \frac{1}{56}\sum_{k=1}^\infty \frac{H_k}{k^7} - \frac{1}{8}\sum_{k=1}^\infty H_k \left(\zeta(8) - \sum_{j=1}^{k} \frac{1}{j^8}\right)$$

To solve the latter summation, we employ Abel's summation formula, namely

$$\sum_{k=1}^\infty a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^\infty A_k (b_k - b_{k+1}),$$

where $A_n = \sum_{k=1}^n a_k$. Choose $a_k = H_k$ and $b_k = \zeta(8) - \sum_{j=1}^k 1/j^8$, and we get that

$$\sum_{k=1}^\infty H_k \left(\zeta(8) - \sum_{j=1}^k \frac{1}{j^8}\right) = \lim_{n\to\infty} (n+1)(H_{n+1}-1)\left(\zeta(8) - \sum_{j=1}^{n+1} \frac{1}{j^8}\right) + \sum_{k=1}^\infty \frac{(k+1)(H_{k+1}-1)}{(k+1)^8}$$

$$= 0 + \sum_{k=1}^\infty \frac{(H_{k+1}-1)}{(k+1)^7} = \sum_{k=1}^\infty \frac{H_k}{k^7} + \sum_{k=1}^\infty \frac{1}{k^7}.$$

In the above calculation, we used the well-known result, $\sum_{k=1}^n H_k = (n+1)(H_{n+1}-1)$. Putting the result of the latter sum back to A gives us

$$A = \frac{1}{56}\sum_{k=1}^\infty \frac{H_k}{k^7} - \frac{1}{8}\sum_{k=1}^\infty \frac{H_k}{k^7} + \frac{1}{8}\sum_{k=1}^\infty \frac{1}{k^7} = -\frac{3}{28}\sum_{k=1}^\infty \frac{H_k}{k^7} + \frac{\zeta(7)}{8}. \tag{1}$$

Using the classical Euler's formula for harmonic sum, we obtain that

$$\sum_{k=1}^\infty \frac{H_k}{k^7} = \frac{\pi^8}{4200} - \zeta(3)\zeta(5)$$

Putting the value in (1), the announced closed form is proved.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\int_1^\infty \frac{\{kx\}}{x^9}dx = \int_0^\infty \frac{\{kx\}}{(x+1)^9}dx = \frac{1}{k}\int_0^\infty \frac{\{x\}}{\left(\frac{x}{k}+1\right)^9}dx = k^8\int_0^\infty \frac{\{x\}}{(x+k)^9}dx = k^8\sum_{n=0}^\infty\int_0^1 \frac{x}{(x+n+k)^9}dx =$$

$$= k^8\sum_{n=0}^\infty\int_0^1 \frac{x+n+k}{(x+n+k)^9}dx - k^8\sum_{n=0}^\infty\int_0^1 \frac{n+k}{(x+n+k)^9}dx =$$

$$= k^8\sum_{n=0}^\infty\left(\frac{1}{7(n+k)^7} - \frac{1}{7(n+k+1)^7}\right) - k^8\sum_{n=0}^\infty\left(\frac{n+k}{8(n+k)^8} - \frac{n+k}{8(n+k+1)^8}\right) =$$

$$= \frac{k}{7} - \frac{k}{8} - k^8\sum_{n=k+1}^\infty \frac{1}{8n^8} = \frac{k}{56} - \frac{1}{8}k^8\sum_{n=k+1}^\infty\frac{1}{n^8}.$$

So

$$\int_1^\infty\left(\frac{\sum_{k=1}^\infty \{kx\}\frac{H_k}{k^8}}{x^9}\right)dx = \sum_{k=1}^\infty\left(\frac{k}{56} - \frac{1}{8}k^8\sum_{n=k+1}^\infty\frac{1}{n^8}\right)\frac{H_k}{k^8} = \sum_{k=1}^\infty H_k\left(\frac{1}{56k^7} - \frac{1}{8}\sum_{n=k+1}^\infty\frac{1}{n^8}\right).$$

We have

$$\sum_{k=1}^\infty \frac{H_k}{k^7} = \frac{\pi^8}{4200} - (3)(5),$$

since by a theorem of Euler

$$\sum_{k=1}^\infty\frac{H_k}{k^n} = \left(1+\frac{n}{2}\right)(n+1) - \frac{1}{2}\sum_{j=1}^{n-2}(j+1)(n-j),$$

if n is an integer $\geqslant 2$. In addition, $(2) = \frac{\pi^2}{6}$, $(4) = \frac{\pi^4}{90}$, $(6) = \frac{\pi^6}{945}$, $(8) = \frac{\pi^8}{9450}$.
Furthermore

$$\frac{1}{(n+1)^8} = -\frac{1}{5040}\int_0^1 x^n\log^7 x\,dx, \qquad \sum_{n=k+1}^\infty\frac{1}{n^8} = -\frac{1}{5040}\int_0^1\frac{x^k}{1-x}\log^7 x\,dx$$

and

$$\sum_{k=1}^\infty H_k x^k = \frac{-\log(1-x)}{1-x}, \quad |x|<1.$$

Hence

$$\sum_{k=1}^\infty H_k\sum_{n=k+1}^\infty\frac{1}{n^8} = -\frac{1}{5040}\sum_{k=1}^\infty H_k\int_0^1\frac{x^k}{1-x}\log^7 x\,dx = \frac{1}{5040}\int_0^1\frac{\log^7 x\log(1-x)}{(1-x)^2}dx == \frac{1}{4200}\pi^8 - (3)(5) - (7).$$

The integral was evaluated with Mathematica by executing the command

$$\text{Integrate}[\text{Log}[x] \wedge 7\text{Log}[1-x]/(1-x) \wedge 2, \{x, 0, 1\}]/5040.$$

Finally,

$$\int_1^\infty \left(\frac{\sum_{k=1}^\infty \{kx\} \frac{H_k}{k^8}}{x^9} \right) dx = \sum_{k=1}^\infty H_k \left(\frac{1}{56k^7} - \frac{1}{8} \sum_{n=k+1}^\infty \frac{1}{n^8} \right) =$$

$$= \frac{1}{56} \left(\frac{\pi^8}{4200} - (3)\,(5) \right) - \frac{1}{8} \left(\frac{1}{4200}\pi^8 - (3)\,(5) - (7) \right) =$$

$$= \frac{(7)}{8} + \frac{3}{28}(3)\,(5) - \frac{\pi^8}{39200}.$$

This completes the prove. To evaluate $\frac{1}{5040} \int_0^1 \frac{\log^7 x \log(1-x)}{(1-x)^2} dx$ by «paper and pencil» (without the help of a computer) we start from Euler's evaluation of the beta function

$$\frac{(u)\,(v)}{(u+v)} = \int_0^1 (1-x)^{u-1} x^{v-1} dx, \;\; Re\,(u) > 0, \; Re\,(v) > 0.$$

We differentiate this equation seven times with respect to v and then set v=1:

$$(u)\,\frac{d^7}{dv^7}\left(\frac{(v)}{(u+v)} \right)\Bigg|_{v=1} = \frac{(u+2)}{u(u+1)} \frac{d^7}{dv^7} \left(\frac{(u+v+1)\,(v+1)}{(u+v+2)} \right)\Bigg|_{v=0} == \int_0^1 (1-x)^{u-1} \log^7 x\, dx, \;\; Re\,(u) > -7.$$

Then

$$\int_0^1 \frac{\log^7 x \log(1-x)}{(1-x)^2} dx = \frac{d}{du} \left(\frac{(u+2)}{u\,(u+1)} \frac{d^7}{dv^7} \left(\frac{(u+v+1)\,(v+1)}{(u+v+2)} \right)\Bigg|_{v=0} \right)\Bigg|_{u=-1} =$$

$$= \frac{d}{du} \left(\frac{(u+1)}{u(u-1)} \frac{d^7}{dv^7} \left(\frac{(u+v)\,(v+1)}{(u+v+1)} \right)\Bigg|_{v=0} \right)\Bigg|_{u=0}.$$

Let $(z+1) = \sum_{k=0}^\infty a_k z^k$ and $\frac{1}{(z+1)} = \sum_{k=0}^\infty b_k z^k$ be the Taylor expansions of Γ(z+1) and 1/Γ(z+1) around z=0. It is known (I.S. Gradshteyn / I.M. Ryzhik, Table of Integrals, Series, and Products, corrected and enlarged edition, Academic Press, 1980, formula 8.321 & 8.363.1) that $a_0 = 1, a_1 = -\gamma$, and

$$\frac{'\,(z+1)}{(z+1)} = -\gamma + \sum_{k=2}^\infty (-1)^k\,(k)\,z^{k-1},$$

$$\frac{''\,(z+1)}{(z+1)} = \frac{d}{dz} \left(\frac{'\,(z+1)}{(z+1)} \right) + \left(\frac{'\,(z+1)}{(z+1)} \right)^2 = \sum_{k=2}^\infty (-1)^k\,(k-1)\,(k)\,z^{k-2} + \left(-\gamma + \sum_{k=2}^\infty (-1)^k\,(k)\,z^{k-1} \right)^2.$$

Therefore

$$\frac{d}{du}\left(\frac{(u+1)}{u(u-1)}\frac{d^7}{dv^7}\frac{(u+v)(v+1)}{(u+v+1)}\bigg|_{v=0}\right)\bigg|_{u=0} =$$

$$= -7!\left[u^1\right]\left[v^7\right]\sum_{h\geqslant 0, i\geqslant 0, j\geqslant 0, k\geqslant 0} u^{h-1}a_i u^i a_j v^j b_k (u+v)^{k+1} =$$

$$= -7!\left[u^1\right]\left[v^7\right]\sum_{h\geqslant 0, i\geqslant 0, j\geqslant 0, m\geqslant 0, n\geqslant 0, m+n\geqslant 1} u^{h-1}a_i u^i a_j v^j b_{m+n-1}\begin{pmatrix} m+n \\ m \end{pmatrix} u^m v^n =$$

$$= -7!\sum_{0\leqslant j\leqslant 7} a_0 a_j b_{8-j}\begin{pmatrix} 9-j \\ 2 \end{pmatrix} - 7!\sum_{0\leqslant i\leqslant 1, 0\leqslant j\leqslant 7} a_i a_j b_{7-j}\begin{pmatrix} 8-j \\ 1 \end{pmatrix} - 7!\sum_{0\leqslant i\leqslant 2, 0\leqslant j\leqslant 6} a_i a_j b_{6-j}\begin{pmatrix} 7-j \\ 0 \end{pmatrix} =$$

$$= -7!\sum_{0\leqslant j\leqslant 8} a_j b_{8-j}\begin{pmatrix} 9-j \\ 2 \end{pmatrix} - 7!\,(1-\gamma)\sum_{0\leqslant j\leqslant 7} a_j b_{7-j}\,(8-j) - 7!\,(a_0+a_1+a_2)\sum_{0\leqslant j\leqslant 6} a_j b_{6-j} =$$

$$= -7!\sum_{0\leqslant j\leqslant 8} a_j b_{8-j}\left(\frac{1}{2}j\,(j-1) - 8j + 36\right) + 7!\,(1-\gamma)\sum_{0\leqslant j\leqslant 7} j a_j b_{7-j} =$$

$$= -7!\left[z^6\right]\frac{''(z+1)}{2\,(z+1)} + 8\cdot 7!\left[z^7\right]\frac{'(z+1)}{(z+1)} + 7!\,(1-\gamma)\left[z^6\right]\frac{'(z+1)}{(z+1)} =$$

$$= -7!\frac{1}{2}\left(7\,(8) +^2 (4) + 2\,(3)\,(5) + 2\,(2)\,(6) + 2\gamma\,(7)\right) + 8\cdot 7!\,(8) - 7!\,(1-\gamma)\,(7) =$$

$$= 7!\left(\frac{1}{4200}\pi^8 - (3)\,(5) - (7)\right),$$

in agreement with the result obtained by the computer.

Also solved by the problem proposer.

• **5744** *Proposed by Toyesh Prakash Sharma (Student) Agra College, India.*

Show that

$$\left(\int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}}\mathrm{dx}\right)^2 + \left(\int_1^\infty \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}}\mathrm{dx}\right)^2 = \frac{\pi}{\sqrt{5}}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Let z>0. Then

$$\int_0^\infty \frac{e^{-zt}}{\sqrt{t}}dt = \frac{1}{\sqrt{z}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}}dt = \frac{1}{\sqrt{z}}\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{z}}.$$

The integral $\int_0^\infty \frac{e^{-zt}}{\sqrt{t}}dt$ converges absolutely in Re(z)>0 and uniformly in Re(z)⩾ϵ>0, and therefore represents an analytic function in Re(z)>0. Hence, by the principle of analytic continuation,

$$\int_0^\infty \frac{e^{-zt}}{\sqrt{t}}dt = \sqrt{\frac{\pi}{z}},\ \ Re\,(z) > 0,$$

where $\sqrt{z}$ is the main branch of the square root defined by $\sqrt{z} = \sqrt{|z|}e^{\frac{1}{2}i\arg z}$, -π<arg z<π.
So

$$\int_1^\infty \frac{\cos\left(\ln x^2\right) + i\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}}dx = \int_1^\infty \frac{e^{i\ln x^2}}{x^2\sqrt{\ln x}}dx \overbrace{=}^{x=e^t} \int_0^\infty \frac{e^{-(1-2i)t}}{\sqrt{t}}dt = \sqrt{\frac{\pi}{1-2i}} = \sqrt{\frac{\pi}{\sqrt{5}}}e^{\frac{1}{2}i\arctan 2}.$$

We separate real and imaginary part and obtain

$$\int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}}dx = \sqrt{\frac{\pi}{\sqrt{5}}}\cos\left(\frac{1}{2}\arctan 2\right),$$

$$\int_1^\infty \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}}dx = \sqrt{\frac{\pi}{\sqrt{5}}}\sin\left(\frac{1}{2}\arctan 2\right).$$

Finally

$$\left(\int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}}dx\right)^2 + \left(\int_1^\infty \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}}dx\right)^2 =$$

$$= \frac{\pi}{\sqrt{5}}\left(\cos^2\left(\frac{1}{2}\arctan 2\right) + \sin^2\left(\frac{1}{2}\arctan 2\right)\right) = \frac{\pi}{\sqrt{5}}.$$

Solution 2 by Ankush Kumar Parcha, Indira Gandhi National Open University, New Delhi, India.

Let

$$\xi_1 = \int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}}\mathrm{dx} \overbrace{\Rightarrow}^{\ln x \longrightarrow x} \int_0^\infty \frac{\cos(2x)}{\sqrt{x}}e^{-x}\mathrm{dx} \Longrightarrow \Re\int_0^\infty \frac{e^{-(1-2i)x}}{\sqrt{x}}\mathrm{dx}.$$

Now using Mellin Transform and $\mathcal{M}\left\{e^{-ax}\right\}(z) = \Gamma(z)/a^z$

$$\Re\int_0^\infty \frac{e^{-(1-2i)x}}{\sqrt{x}}\mathrm{dx} = \Re\left\{\mathcal{M}\left\{e^{-(1-2i)x}\right\}\left(\frac{1}{2}\right)\right\} = \Re\left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{1-2i}}\right\} = \sqrt{\pi}\,\Re\left\{\sqrt{\frac{1+2i}{5}}\right\}.$$

As $\ln(x+iy) = \frac{1}{2}\ln\left(x^2+y^2\right) + i\arctan\left(\frac{y}{x}\right)$, then

$$\sqrt{\pi}\,\Re\left\{\sqrt{\frac{1+2i}{5}}\right\} = \sqrt{\frac{\pi}{5}}\Re\left\{\sqrt[4]{5}e^{i\frac{\arctan 2}{2}}\right\} = \sqrt{\frac{\pi}{5}}\Re\left\{\sqrt[4]{5}\left(\cos\left(\frac{\arctan 2}{2}\right) + i\sin\left(\frac{\arctan 2}{2}\right)\right)\right\}$$

$$= \sqrt{\frac{\pi}{\sqrt{5}}} \cos\left(\frac{\arctan 2}{2}\right).$$

Similarly

$$\xi_1 = \int_1^\infty \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}} \mathrm{dx} = \sqrt{\frac{\pi}{\sqrt{5}}} \sin\left(\frac{\arctan 2}{2}\right).$$

Then

$$\left(\int_1^\infty \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}} \mathrm{dx}\right)^2 + \left(\int_1^\infty \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}} \mathrm{dx}\right)^2 = \left(\sqrt{\frac{\pi}{\sqrt{5}}} \cos\left(\frac{\arctan 2}{2}\right)\right)^2 + \left(\sqrt{\frac{\pi}{\sqrt{5}}} \sin\left(\frac{\arctan 2}{2}\right)\right)^2 = \frac{\pi}{\sqrt{\ }}$$

Hence the equality is proved.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Consider the more general problem of evaluating

$$\left(\int_1^\infty \frac{\cos(\ln x^\alpha)}{x^2\sqrt{\ln x}}\, dx\right)^2 + \left(\int_1^\infty \frac{\sin(\ln x^\alpha)}{x^2\sqrt{\ln x}}\, dx\right)^2$$

for some real number α. Let

$$I = \int_1^\infty \frac{\cos(\ln x^\alpha)}{x^2\sqrt{\ln x}}\, dx \quad \text{and} \quad J = \int_1^\infty \frac{\sin(\ln x^\alpha)}{x^2\sqrt{\ln x}}\, dx.$$

With the substitution $u = \ln x$,

$$I = \int_0^\infty u^{-1/2} e^{-u} \cos(\alpha u)\, du = \operatorname{Re} \int_0^\infty u^{-1/2} e^{-(1-i\alpha)u}\, du$$

and

$$J = \int_0^\infty u^{-1/2} e^{-u} \sin(\alpha u)\, du = \operatorname{Im} \int_0^\infty u^{-1/2} e^{-(1-i\alpha)u}\, du.$$

Now,

$$\int_0^\infty u^{-1/2} e^{-(1-i\alpha)u}\, du = \Gamma\left(\frac{1}{2}\right)(1 - i\alpha)^{-1/2} = \frac{\sqrt{\pi}}{\sqrt[4]{1+\alpha^2}} e^{\frac{1}{2} i \tan^{-1}\alpha},$$

so

$$I = \frac{\sqrt{\pi}}{\sqrt[4]{1+\alpha^2}} \cos\left(\frac{1}{2}\tan^{-1}\alpha\right), \quad J = \frac{\sqrt{\pi}}{\sqrt[4]{1+\alpha^2}} \sin\left(\frac{1}{2}\tan^{-1}\alpha\right),$$

and

$$\left(\int_1^\infty \frac{\cos(\ln x^\alpha)}{x^2\sqrt{\ln x}}\, dx\right)^2 + \left(\int_1^\infty \frac{\sin(\ln x^\alpha)}{x^2\sqrt{\ln x}}\, dx\right)^2 = I^2 + J^2 = \frac{\pi}{\sqrt{1+\alpha^2}}.$$

In particular, with $\alpha = 2$,

$$\left(\int_1^\infty \frac{\cos(\ln x^2)}{x^2\sqrt{\ln x}}\,dx\right)^2 + \left(\int_1^\infty \frac{\sin(\ln x^2)}{x^2\sqrt{\ln x}}\,dx\right)^2 = \frac{\pi}{\sqrt{5}}.$$

Solution 4 by G. C. Greubel, Newport News, VA .

First consider the integral:

$$J_1 = \int_1^\infty \frac{\cos(\ln(x^2))}{x^2\ \sqrt{\ln x}}\,dx$$

which can be evaluated by making use of the change of variable $u = \ln x$ which leads to

$$\begin{aligned} J_1 &= \int_1^\infty \frac{\cos(2\ \ln x)}{x^2\ \sqrt{\ln x}}\,dx \\ &= \int_0^\infty e^{-u}\ \cos(2u)\,u^{-1/2}\,du \\ &= \frac{1}{2}\int_0^\infty \left(e^{-(1+2i)\,u} + e^{-(1-2i)\,u}\right)\ u^{-1/2}\,du \\ &= \frac{1}{2}\sqrt{\frac{\pi}{5}}\,(\sqrt{1+2i} + \sqrt{1-2i}). \end{aligned}$$

The second integral follows the same pattern and is

$$J_2 = \int_1^\infty \frac{\sin(\ln(x^2))}{x^2\ \sqrt{\ln x}}\,dx = \frac{-i}{2}\sqrt{\frac{\pi}{5}}\,(\sqrt{1+2i} - \sqrt{1-2i}).$$

Now consider a more general problem in the form

$$I_p = \left(\int_1^\infty \frac{\cos(\ln(x^2))}{x^2\ \sqrt{\ln x}}\,dx\right)^p + \left(\int_1^\infty \frac{\sin(\ln(x^2))}{x^2\ \sqrt{\ln x}}\,dx\right)^p.$$

With the evaluated integrals then

$$I_p = \frac{1}{2^p}\left(\frac{\pi}{5}\right)^{p/2}\left((\sqrt{1+2i} + \sqrt{1-2i})^p + (-i)^p\,(\sqrt{1+2i} - \sqrt{1-2i})^p\right).$$

This is the general result desired. To reveal some interesting cases first consider setting $p \to 2p$

which yields

$$
\begin{aligned}
I_{2p} &= \frac{1}{4^p}\left(\frac{\pi}{5}\right)^p \left((\sqrt{1+2i}+\sqrt{1-2i})^{2p}+(-1)^p(\sqrt{1+2i}-\sqrt{1-2i})^{2p}\right)\\
&= \frac{1}{2^p}\left(\frac{\pi}{5}\right)^p \left((1+\sqrt{5})^p+(-1)^p(1-\sqrt{5})^p\right)\\
&= \left(\frac{\pi}{5}\right)^p \left(\alpha^p+(-1)^p\beta^p\right)\\
&= \frac{1}{2}\left(\frac{\pi}{5}\right)^p \left((L_p+\sqrt{5}\,F_p)+(-1)^p(L_p-\sqrt{5}\,F_p)\right)\\
&= \frac{1}{2}\left(\frac{\pi}{5}\right)^p \left((1+(-1)^p)*L_p+\sqrt{5}\,(1-(-1)^p)\,F_p)\right),
\end{aligned}
$$

where $2\,\alpha = 1+\sqrt{5}$, $2\beta = 1-\sqrt{5}$, L_p and F_p are the Lucas and Fibonacci numbers, respectively. Now setting $p \to 2p$ and $p \to 2p+1$ yields

$$
\begin{aligned}
I_{4p} &= \left(\frac{\pi}{5}\right)^{2p} L_{2p}\\
I_{4p+2} &= \frac{\pi}{\sqrt{5}}\left(\frac{\pi}{5}\right)^{2p} F_{2p+1}.
\end{aligned}
$$

Another interesting relation can be obtained by using $n!\,\zeta(n) = 2^{n-1}\,|B_n|\,\pi^n$ which yields

$$
\begin{aligned}
I_{4p} &= \frac{2\,\Gamma(2p+1)\,\zeta(2p)\,L_{2p}}{(10)^{2p}\,|B_{2p}|}\\
I_{4p+2} &= \frac{\pi}{\sqrt{5}}\,\frac{2\,\Gamma(2p+1)\,\zeta(2p)\,F_{2p+1}}{(10)^{2p}\,|B_{2p}|}.
\end{aligned}
$$

These last two formulas combine the Gamma function, Zeta function, Fibonacci and Lucas numbers, and Bernoulli numbers into the result.

Returning to the proposed problem set $p = 0$ in

$$
I_{4p+2} = \frac{\pi}{\sqrt{5}}\left(\frac{\pi}{5}\right)^{2p} F_{2p+1}
$$

to obtain

$$
I_2 = \frac{\pi}{\sqrt{5}}
$$

which is the desired result of the proposed problem.

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

The substitution $x = e^{t^2/2}$ transforms the cosine and sine integrals into

$$\sqrt{2}\int_0^\infty e^{-t^2/2}\cos(t^2)\,dt \quad \text{and} \quad \sqrt{2}\int_0^\infty e^{-t^2/2}\sin(t^2)\,dt, \text{ respectively.} \tag{2}$$

In a *College Mathematics Journal* article (Vol. 40 (2009), No. 4), Hongwei Chen proves the following formulas for these Fresnel-like integrals

$$\int_0^\infty e^{-tx^2}\cos(px^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{t+\sqrt{t^2+p^2}}{t^2+p^2}}$$
$$\int_0^\infty e^{-tx^2}\sin(px^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{-t+\sqrt{t^2+p^2}}{t^2+p^2}}.$$

With $t = 1/2$ and $p = 1$, we apply these results to the integrals in (1) to get

$$\sqrt{2}\int_0^\infty e^{-t^2/2}\cos(t^2)\,dt = \frac{\sqrt{\pi}\sqrt{10+10\sqrt{5}}}{10}$$
$$\sqrt{2}\int_0^\infty e^{-t^2/2}\sin(t^2)\,dt = \frac{\sqrt{\pi}\sqrt{-10+10\sqrt{5}}}{10},$$

which implies the desired result after some routine algebra.

Solution 6 by Michel Bataille, Rouen, France.

The change of variables $x = e^t$ gives

$$A := \int_1^\infty \frac{\cos(\ln x^2)}{x^2\sqrt{\ln x}}\,dx = \int_0^\infty t^{-1/2}e^{-t}\cos(2t)\,dt$$

and

$$B := \int_1^\infty \frac{\sin(\ln x^2)}{x^2\sqrt{\ln x}}\,dx = \int_0^\infty t^{-1/2}e^{-t}\sin(2t)\,dt$$

so that

$$A^2 + B^2 = |A + iB|^2 = \left|\int_0^\infty t^{-1/2}e^{-t(1-2i)}\,dt.\right|^2$$

We know that if $\alpha > -1$, then the Laplace transform of t^α is $\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ $(Re(s) > 0)$. It follows that

$$A + iB = \int_0^\infty t^{-1/2}e^{-t(1-2i)}\,dt = \frac{\Gamma(1/2)}{(1-2i)^{1/2}}$$

and

$$A^2 + B^2 = \frac{(\Gamma(1/2))^2}{|(1-2i)^{1/2}|^2} = \frac{\pi}{\sqrt{5}}$$

(since $\Gamma(1/2 = \sqrt{\pi}$ and with the principal determination of $z^{1/2}$).

Note that $\int_0^{\infty} e^{-st} t^{\alpha}\, dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ for $\alpha > -1$ and $Re(s) > 0$ can also be obtained as follows: it is easily checked when s is a positive real number (change of variables $u = st$); since $s \mapsto \int_0^{\infty} e^{-st} t^{\alpha}\, dt$ is ananlytic in $Re(s) > 0$, the result follows by analytic extension.

Solution 7 by Péter Fülöp, Gyömrő, Hungary.

After performing the $t = \ln x$ substituion at both integrals, we get:

$$I_1 = \int_0^{\infty} \frac{\cos(2t)}{e^{2t}\sqrt{t}} e^t dt = \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} e^{(-1+2i)t} dt + \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} e^{(-1-2i)t} dt$$

Similar to the second integral:

$$I_2 = \int_0^{\infty} \frac{\sin(2t)}{e^{2t}\sqrt{t}} e^t dt = \frac{1}{2i}\int_0^{\infty} t^{-\frac{1}{2}} e^{(-1+2i)t} dt - \frac{1}{2i}\int_0^{\infty} t^{-\frac{1}{2}} e^{(-1-2i)t} dt$$

Let's apply the $u = (1-2i)t$ substitution to the first integrals of I_1 and I_2, and apply the $u = (1+2i)t$ substitution to the second ones of I_1 and I_2, we get:

$$I_1 = \frac{1}{2}\int_0^{\infty} (1-2i)^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-u} du + \frac{1}{2}\int_0^{\infty} (1+2i)^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-u} du$$

$$I_2 = \frac{1}{2i}\int_0^{\infty} (1-2i)^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-u} du - \frac{1}{2i}\int_0^{\infty} (1+2i)^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-u} du$$

Since $\int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, the values of I_1 and I_2 are equal to the following complex numbers:

$$I_1 = \frac{1}{2}(1-2i)^{-\frac{1}{2}}\sqrt{\pi} + \frac{1}{2}(1+2i)^{-\frac{1}{2}}\sqrt{\pi}$$

$$I_2 = \frac{1}{2i}(1-2i)^{-\frac{1}{2}}\sqrt{\pi} - \frac{1}{2i}(1+2i)^{-\frac{1}{2}}\sqrt{\pi}$$

Finally we can calculate the value of $I_1^2 + I_2^2$

$$I_1^2 + I_2^2 =$$
$$\frac{\pi}{4}\Big((1-2i)^{-1} + (1+2i)^{-1} + 2(1-2i)^{-\frac{1}{2}}(1+2i)^{-\frac{1}{2}}\Big) - \frac{\pi}{4}\Big((1-2i)^{-1} + (1+2i)^{-1} - 2(1-2i)^{-\frac{1}{2}}(1+2i)^{-\frac{1}{2}}\Big)$$

$$I_1^2 + I_2^2 = \frac{\pi}{4} 4[\underbrace{(1-2i)(1+2i)}_{5}]^{-\frac{1}{2}} = \underline{\frac{\pi}{\sqrt{5}}}$$

Solution 8 by Srikanth Pai, International Centre for Theoretical Sciences (ICTS-TIFR), Bangalore, India.

Let us denote the integral on the right by I. Now consider the integral

$$J = \int_1^\infty \frac{\exp(i \ln x^2)}{x^2 \sqrt{\ln x}}\, dx = \left(\frac{\cos(\ln x^2)}{x^2 \sqrt{\ln x}}\right) + i \left(\frac{\cos(\ln x^2)}{x^2 \sqrt{\ln x}}\right).$$

Note that $I = |J|^2$ by linearity of the integral operation. If we substitute $\ln x = u^2$, then we can use $\dfrac{dx}{x} = 2u\, du$ to obtain:

$$J = \int_0^\infty 2e^{-u^2} e^{i2u^2} du$$

Using a change of variables $w = u\sqrt{1-2i}$, we get

$$I = |J|^2 = \left| \int_0^\infty 2e^{-w^2} \frac{dw}{\sqrt{1-2i}} \right|^2 = \frac{\pi}{\sqrt{5}}$$

In the last step, we have used the fact that $|1-2i|^2 = 5$ and the standard Gaussian integral

$$\int_{-\infty}^\infty e^{-w^2} = \sqrt{\pi}.$$

Solution 9 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $t = \sqrt{\ln x}, \quad t^2 = \ln x, \quad x = e^{t^2}, \quad \mathrm{d}x = x2t\mathrm{d}t$

$$\int_1^\infty \frac{\cos(\ln x^2)}{x^2 \sqrt{\ln x}} \mathrm{d}x = \int_0^\infty \frac{\cos(2t^2)}{e^{t^2} t} 2t\mathrm{d}t = \int_0^\infty \frac{2\cos(2t^2)}{e^{t^2}} \mathrm{d}t$$

$$2e^{-t^2}\cos(2t^2) = e^{-(1-2i)t^2} + e^{-(1+2i)t^2}$$

$$\because \quad \int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}, \quad \int_0^\infty e^{-zx^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{z}}$$

$$\therefore \quad \int_0^\infty \frac{2\cos(2t^2)}{e^{t^2}}\mathrm{d}t = \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{1-2i}} + \frac{1}{\sqrt{1+2i}}\right) = \frac{\sqrt{\pi}}{2} \frac{\sqrt{1+2i} + \sqrt{1-2i}}{\sqrt{5}}$$

$$\because \quad 1 \pm 2i = \sqrt{5}e^{\pm i\theta}, \quad \theta = \arctan 2$$

$$\therefore \quad \sqrt{1+2i}+\sqrt{1-2i} = \sqrt[4]{5}\left(e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}\right) = 2\sqrt[4]{5}\cos\frac{\theta}{2} = 2\sqrt[4]{5}\sqrt{\frac{1+\cos\theta}{2}} = 2\sqrt[4]{5}\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}}$$

$$\therefore \quad \int_0^\infty \frac{2\cos(2t^2)}{e^{t^2}}\mathrm{d}t = \frac{\sqrt{\pi}}{2}\times\frac{1}{\sqrt{5}}\times 2\times\sqrt[4]{5}\times\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}} = \sqrt{\frac{(1+\sqrt{5})\pi}{10}}$$

Similarly

$$\int_1^\infty \frac{\sin(\ln x^2)}{x^2\sqrt{\ln x}}\mathrm{d}x = \int_0^\infty \frac{2\sin(2t^2)}{e^{t^2}}\mathrm{d}t = \frac{\sqrt{\pi}}{\sqrt[4]{5}}\sin\left(\frac{\arctan 2}{2}\right) = \sqrt{\frac{(\sqrt{5}-1)\pi}{10}}$$

$$\therefore \quad \left(\int_1^\infty \frac{\cos(\ln x^2)}{x^2\sqrt{\ln x}}\mathrm{d}x\right)^2 + \left(\int_1^\infty \frac{\sin(\ln x^2)}{x^2\sqrt{\ln x}}\mathrm{d}x\right)^2 = \frac{(\sqrt{5}+1)\pi}{10}+\frac{(\sqrt{5}-1)\pi}{10} = \frac{\sqrt{5}\pi}{5} = \frac{\pi}{\sqrt{5}}$$

Q.E.D

Appendix

$$\int_0^\infty \frac{2\sin(2t^2)}{e^{t^2}}\mathrm{d}t = \int_0^\infty 2e^{-t^2}\sin(2t^2)\mathrm{d}t = \int_0^\infty ie^{-t^2}\left(e^{-2it^2}-e^{2it^2}\right)\mathrm{d}t$$

$$= i\int_0^\infty \left(e^{-(1+2i)t^2}-e^{(1+2i)t^2}\right)\mathrm{d}t = i\frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{1+2i}}-\frac{1}{\sqrt{1-2i}}\right) = \frac{i}{\sqrt{5}}\times\frac{\sqrt{\pi}}{2}\left(\sqrt{1-2i}-\sqrt{1+2i}\right)$$

$$= \frac{i}{\sqrt{5}}\times\frac{\sqrt{\pi}}{2}\times\sqrt[4]{5}\left(e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}\right) = -\frac{2\sqrt[4]{5}i}{\sqrt{5}}\times\sin\frac{\theta}{2}\times\frac{i\sqrt{\pi}}{2}, \quad \theta = \arctan 2$$

$$\therefore \quad \int_0^\infty \frac{2\sin(2t^2)}{e^{t^2}}\mathrm{d}t = \sqrt{\pi}\times\frac{\sqrt[4]{5}}{\sqrt{5}}\times\sin\frac{\arctan 2}{2} = \frac{\sqrt{5}}{\sqrt[4]{5}}\sin\frac{\arctan 2}{2}$$

$$\because \quad \sin^2\left(\frac{\arctan 2}{2}\right) = \frac{1-\cos(\arctan 2)}{2} = \frac{1-\frac{1}{\sqrt{5}}}{2} = \frac{\sqrt{5}-1}{\sqrt{20}}$$

$$\therefore \quad \sin\frac{\arctan 2}{2} = \sqrt{\frac{\sqrt{5}-1}{\sqrt{20}}}$$

$$\therefore \quad \frac{\sqrt{\pi}}{\sqrt[4]{5}}\sin\frac{\arctan 2}{2} = \frac{\sqrt{\pi}}{\sqrt[4]{5}}\sqrt{\frac{\sqrt{5}-1}{\sqrt{20}}} = \sqrt{\pi}\sqrt{\frac{\sqrt{5}-1}{\sqrt{20}\sqrt{5}}} = \sqrt{\frac{(\sqrt{5}-1)\pi}{10}}.$$

Also solved by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal; and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣