Problems and Solutions Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please present all proposed solutions and proposed problems according to formatting requirements delineated near the end of this document. Also, please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

To propose problems, email them to: problems4ssma@gmail.com

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Solutions to the problems published in this issue should be submitted *before* December 1, 2024.

• **5781** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania..*

Let $m, n, p, q, r, s \in \mathbb{N}\backslash\{0\}$ and define

$$H_n^{(m)} = \frac{1}{1^m} + \frac{1}{2^m} + \ldots + \frac{1}{n^m}.$$

Prove that

$$\left(H_n^{(2p)} + H_n^{(2q)}\right)\left(H_n^{(2r)} + H_n^{(2s)}\right) \geqslant \left(H_n^{(p+r)} + H_n^{(q+s)}\right)^2.$$

• **5782** *Proposed by Toyesh Prakash Sharma and Etisha Sharma, Agra College, Agra, India..*

If $a, b, c \geqslant 1$, then prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geqslant \frac{a+b+c}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right).$$

• **5783** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1, \ldots, x_n > 0$ be real numbers and set $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n} (1+x_i)^{x_i}.$$

When does equality occur?

• **5784** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all postive integers n for which there exist n pairwise-distinct positive integers $x_1, x_2, ..., x_n$ satisfying the equation:

$$\ln x_1 + \ln x_2 + \cdots + \ln x_n \; = \; \ln(x_1 + x_2 + \cdots + x_n).$$

where ln denotes natural logarithm.

• **5785** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Prove that 3 is the largest positive value of the constant k such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 \geqslant k(a + b + c + d - 4)$$

for any positive real numbers a, b, c, d with $a \geqslant b \geqslant c \geqslant 1 \geqslant d$ and $ab + bc + cd + da = 4$.

• **5786** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$:

$$f(-x) = 1 - 2\int_0^x \mathrm{e}^{-t} f(x-t)\mathrm{d}t.$$

Solutions

To Formerly Published Problems

• **5757** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $\displaystyle\int_0^1 f(x)dx = 1/2$. Show that

$$2 + \int_0^1 f^2(x)dx \geqslant 6\int_0^1 xf(x)dx.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We prove the more general inequality

$$\frac{1}{4}\int_0^1 f^2(x)\,dx + 2\left(\int_0^1 f(x)\,dx\right)^2 \geqslant 3\int_0^1 f(x)\,dx \cdot \int_0^1 xf(x)\,dx. \tag{1}$$

Then substituting the given integral value and clearing fractions gives us the desired inequality. Now set $\int_0^1 f(x)\,dx = t$ and consider the quadratic polynomial

$$t^2 \;-\; 3\left(\int_0^1 (x-1/3)f(x)\,dx\right) t \;+\; \frac{1}{4}\int_0^1 f^2(x)\,dx. \tag{2}$$

The discriminant of this polynomial is

$$D = 9\left(\int_0^1 (x-1/3)f(x)\,dx\right)^2 - \int_0^1 f^2(x)\,dx.$$

The CBS inequality yields

$$\begin{aligned} D \;&\leqslant\; 9\cdot\int_0^1 (x-1/3)^2\,dx\cdot\int_0^1 f^2(x)\,dx - \int_0^1 f^2(x)\,dx \\ &=\; \int_0^1 f^2(x)\,dx - \int_0^1 f^2(x)\,dx = 0. \end{aligned}$$

Since $D \leqslant 0$ and the coefficient of t^2 in (2) is positive, we see that the quadratic is nonnegative for all values of t. Therefore

$$\begin{aligned} \left(\int_0^1 f(x)\,dx\right)^2 + \frac{1}{4}\int_0^1 f^2(x)\,dx \;&\geqslant\; 3\left(\int_0^1 (x-1/3)f(x)\,dx\right)\cdot\int_0^1 f(x)\,dx \\ &=\; 3\left(\int_0^1 xf(x)\,dx - \frac{1}{3}\int_0^1 f(x)\,dx\right)\cdot\int_0^1 f(x)\,dx \\ &=\; 3\int_0^1 xf(x)\,dx\cdot\int_0^1 f(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2, \end{aligned}$$

which gives us (1).

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

$$\int_0^1 (f-3x+a)^2 dx = \int_0^1 (f^2-6xf+9x^2+a^2+2af-6xa)dx \geqslant 0.$$

Thus

$$\int_0^1 (f^2-6xf)dx \geqslant -3-a^2-a+3a \geqslant -2 \iff (a-1)^2 \leqslant 0 \iff a=1$$

and this concludes the proof.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

Suppose f: [0,1]→ℝ is continuous and $\int_0^1 f(x)\,dx = \frac{1}{2}$. Show that

$$2+\int_0^1 f^2(x)\,dx \geqslant 6\int_0^1 xf(x)\,dx.$$

Solution of the problem
We have

$$0 \leqslant \int_0^1 \left(f(x)-3x+1\right)^2 dx = \int_0^1 \left(f^2(x)+9x^2+1-6xf(x)+2f(x)-6x\right)dx =$$

$$= \int_0^1 f^2(x)\,dx + 3 + 1 - 6\int_0^1 xf(x)\,dx + 1 - 3$$

which implies

$$2+\int_0^1 f^2(x)\,dx \geqslant 6\int_0^1 xf(x)\,dx.$$

Solution 4 by Moti Levy, Rehovot, Israel.

Let $F(x) := \int_0^x f(t)\,dt$. After integration by parts,

$$\int_0^1 xf(x)\,dx = \left. xF(x)\right]_0^1 - \int_0^1 F(x)\,dx = \frac{1}{2} - \int_0^1 F(x)\,dx. \tag{3}$$

Substituting (3) in the original inequality we get

$$2+\int_0^1 \left(F'(x)\right)^2 dx \geqslant 3 - 6\int_0^1 F(x)\,dx \int_0^1 xf(x)\,dx,$$

or

$$\int_0^1 \left(6F(x)+\left(F'(x)\right)^2\right)dx \geqslant 1.$$

Let

$$J(F) := \int_0^1 \left(6F(x)+\left(F'(x)\right)^2\right)dx \geqslant 1,$$

then the original inequality is equivalent to the statement that the functional $J(F)$ is greater than or equal to 1 for every differentiable function $F(x)$, which satisfies the boundary conditions $F(0)=0$ and $F(1)=\frac{1}{2}$.

Every differentiable function $F(x)$, which satisfies the boundary conditions $F(0)=0$ and $F(1)=\frac{1}{2}$ can be expressed as $F(x)=\frac{3}{2}x^2-x+\eta(x)$, where $\eta(x)$ is differentiable function in the interval $(0,1)$ and $\eta(0)=\eta(1)=0$.

Then

$$J\left(\frac{3}{2}x^2 - x + \eta(x)\right) = \int_0^1 \left(6\left(\frac{3}{2}x^2 - x + \eta(x)\right) + \left(3x - 1 + \eta'(x)\right)^2\right) dx$$

$$= \int_0^1 \left(6\left(\frac{3}{2}x^2 - x\right) + (3x-1)^2\right) dx + \int_0^1 6\eta(x) + 2(3x-1)\eta'(x) + \left(\eta'(x)\right)^2 dx$$

$$= 1 + \int_0^1 6\eta(x) + 2(3x-1)\eta'(x) + \left(\eta'(x)\right)^2 dx$$

Applying integration by parts, we obtain

$$J\left(\frac{3}{2}x^2 - x + \eta(x)\right) = 1 + \int_0^1 \left(\eta'(x)\right)^2 dx.$$

It follows that $J\left(F(x)\right) \geqslant 1$ for every differentiable function $F(x)$ which satisfies $F(0) = 0$ and $F(1) = \frac{1}{2}$. The functional $J(F)$ attains its minimum when $\eta'(x) = 0$ which implies (together with the boundary conditions $\eta(0) = \eta(1) = 0$ that $\eta(x) = 0$ in $(0,1)$.

Solution 5 by Michel Bataille, Rouen, France.

Let $I = \int_0^1 (3x-1)f(x)\,dx$. Then, we have

$$6\int_0^1 xf(x)\,dx = 2I + 2\int_0^1 f(x)\,dx = 2I + 1.$$

On the other hand, since $\int_0^1 (3x-1)^2\,dx = \int_0^1 (9x^2 - 6x + 1)\,dx = 1$, the Cauchy-Schwarz inequality gives

$$\int_0^1 f^2(x)\,dx = \left(\int_0^1 (3x-1)^2\,dx\right)\left(\int_0^1 f^2(x)\,dx\right) \geqslant \left(\int_0^1 (3x-1)f(x)\,dx\right)^2 = I^2.$$

As a result, we obtain

$$2 + \int_0^1 f^2(x)\,dx - 6\int_0^1 xf(x)\,dx \geqslant 2 + I^2 - 2I - 1 = (I-1)^2 \geqslant 0$$

and the desired inequality follows.

Also solved by Michael Brozinsky, Central Islip, NY; Yunyong Zhang, Chinaunicom, Yunnan, China; and the problem proposer.

• **5758** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose $P, Q \in Int(\triangle ABC)$ such that $\beta\overrightarrow{AB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = 0$ and $\overrightarrow{AQ} + \alpha\overrightarrow{QB} + \overrightarrow{BC} = 0$ with $\alpha, \beta, \gamma \in \mathbb{R}; \alpha, \beta \neq 1$. Prove that A, P, Q are collinear if and only if $\alpha + \gamma = \beta + 1$.

Solution 1 by proposed by Hong Biao Zeng, Fort Hays State University, KS.

By condition $\overrightarrow{AQ} + \alpha\overrightarrow{QB} + \overrightarrow{BC} = 0$, we have

$$\overrightarrow{AQ} = -\overrightarrow{BC} - \alpha\overrightarrow{QB} = -\overrightarrow{BP} - \overrightarrow{PC} - \alpha(\overrightarrow{AB} - \overrightarrow{AQ})$$

Hence,

$$\begin{aligned}(1-\alpha)\overrightarrow{AQ} &= -\overrightarrow{BP} - \overrightarrow{PC} - \alpha\overrightarrow{AB} \\ &= (\gamma - 1)\overrightarrow{AP} - (\gamma - 1)(\overrightarrow{AB} + \overrightarrow{BP}) - \overrightarrow{BP} - \overrightarrow{PC} - \alpha\overrightarrow{AB} \\ &= (\gamma - 1)\overrightarrow{AP} + (1 - \gamma - \alpha)\overrightarrow{AB} - \gamma\overrightarrow{BP} - \overrightarrow{PC} \\ &= (\gamma - 1)\overrightarrow{AP} + (1 - \gamma - \alpha + \beta)\overrightarrow{AB}\end{aligned}$$

The last step above used the condition $\beta\overrightarrow{AB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = 0$. Now it is clear that A, P, Q are collinear if and only if the coefficient of $\overrightarrow{AB}$ is zero, i.e. $1 - \gamma - \alpha + \beta = 0$, which is $\alpha + \gamma = \beta + 1$.

Solution 2 by proposed by Albert Stadler, Herrliberg, Switzerland.

We identify the points A, B, C, P, Q of the Euclidean plane with complex numbers of the complex number plane and denote them by a, b, c, p, q, respectively. We assume without loss of generality that a=0. Then the given conditions read as $\beta b + \gamma(p - b) + c - p = 0$ and $q + \alpha(b - q) + c - b = 0$. We solve for p and q and find

$$p = \frac{(\beta - \gamma)b + c}{1 - \gamma}, \quad q = \frac{(1 - \alpha)b - c}{1 - \alpha}.$$

Clearly, p≠0, q≠0, since P, Q ∈ Int(ΔABC) and therefore P≠A and Q≠A.
If z is a complex number we denote by $\overline{z}$ the complex conjugate of z.
Note that $b\overline{c}$ is real if and only if A, B, C are collinear which means that the triangle ABC is degenerate with no interior points. But the triangle has interior points, since P, Q ∈ Int(ΔABC). So $b\overline{c}$ is a complex, non-real number.
A, P, Q are collinear if and only if $\dfrac{p}{q} = \dfrac{1 - \alpha}{1 - \gamma} \cdot \dfrac{(\beta - \gamma)b + c}{(1 - \alpha)b - c}$ is real which is the case if and only if

$$\left((\beta - \gamma)b + c\right)\left((1 - \alpha)\overline{b} - \overline{c}\right)$$

is real and this number is real if and only if

$$(1 - \alpha)\overline{b}c - (\beta - \gamma)b\overline{c} = (1 - \alpha)\left(\overline{b}c + b\overline{c}\right) - (\beta - \gamma + 1 - \alpha)b\overline{c}$$

is real. Note that $\left(\bar{b}c + b\bar{c}\right)$ is real and $b\bar{c}$ is non-real. So above term is real if and only if $\beta - \gamma + 1 - \alpha = 0$. So A, P, Q are collinear if and only if $\alpha+\gamma=\beta+1$.

Solution 3 by proposed by Michel Bataille, Rouen, France.

From $\beta\overrightarrow{AP} + \beta\overrightarrow{PB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = \overrightarrow{0}$, we deduce $\beta\overrightarrow{PA} + (\gamma - \beta)\overrightarrow{PB} - \overrightarrow{PC} = \overrightarrow{0}$, hence $P = (\beta : \gamma - \beta : -1)$ in barycentric coordinates relatively to (A, B, C). Similarly, we easily obtain that $Q = (1 : 1 - \alpha : -1)$. It follows that A, P, Q are collinear if and only if

$$\begin{vmatrix} 1 & \beta & 1 \\ 0 & \gamma - \beta & 1 - \alpha \\ 0 & -1 & -1 \end{vmatrix} = 0,$$

that is, $-(\gamma - \beta) + (1 - \alpha) = 0$ or $\alpha + \gamma = \beta + 1$, as required.

Notes. 1/ The condition for the existence of the points P, Q is $\alpha, \gamma \neq 1$; the value of β does not matter.
2/ The hypothesis $P, Q \in Int(\Delta ABC)$ should be deleted: since $\overrightarrow{BQ} = \dfrac{1}{\alpha - 1}\overrightarrow{AC}$, the point Q is on the parallel to AC through B and cannot be interior to ΔABC.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; and the problem proposer.

• **5759** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all pairs (a, b) of non-negative integers satisfying the equation $a^b - b^a = a + b$.

Solution 1 by proposed by Michel Bataille, Rouen, France.

It is readily checked that $(0, 0), (1, 0)$ and $(2, 5)$ are solutions for (a, b) (agreeing that $x^0 = 1$ for all real x). We show that there are no other solutions. Clearly, no pair $(0, b)$ with $b \geqslant 1$ is a solution and a pair $(a, 0)$ with $a \geqslant 1$ is a solution only if $a = 1$. There remains to show that if (a, b) is a solution with $a, b \geqslant 1$, then $(a, b) = (2, 5)$.
It is readily seen that $a^b - b^a \neq a + b$ for all pairs $(1, n), (n, 1), (n, n), (2, 3), (2, 4)$ where $n \in \mathbb{N}$ and, by an easy induction, that $2^n > n^2 + n + 2$ for all integer $n \geqslant 6$. Also, if $3 \leqslant b < a$, then $\dfrac{\ln a}{a} < \dfrac{\ln b}{b}$; hence $a^b < b^a$ and therefore $a^b - b^a \neq a + b$. The proof will be complete if we show that

$$a^b > b^a + a + b \tag{1}$$

holds for any integers a, b with $a \geqslant 3,\ b \geqslant a + 1$.
We prove (1) by induction on b (for an arbitrary fixed a). We will use the following well-known result:

$$\left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} \leqslant \left(1 + \frac{1}{t}\right)^t < e \quad \text{for all } t \in [3, \infty).$$

For $b = a+1$, (1) writes as $a^{a+1} > (a+1)^a + 2a + 1$ or $a > \left(1+\dfrac{1}{a}\right)^a + u(a)$ where $u(a)$ denotes $\dfrac{2a+1}{a^a}$. Since for $n \geqslant 3$, $\dfrac{u(n+1)}{u(n)} = \dfrac{2n+3}{(n(2n+3)+1)(1+1/n)^n} \leqslant \dfrac{27}{64} < 1$, the sequence $(u(n))_{n\geqslant 3}$ is decreasing and so $u(a) \leqslant u(3) = \dfrac{7}{27}$. Thus

$$\left(1+\frac{1}{a}\right)^a + u(a) < e + \frac{7}{27} < 3 \leqslant a$$

and (1) holds for $b = a+1$.
As for the inductive step, if (1) holds for some $b \geqslant a+1$, then

$$a^{b+1} = a.a^b > ab^a + a^2 + ab > (b+1)^a + a + b + 1$$

the latter inequality since

$$ab^a - (b+1)^a = b^a\left(a - \left(1+\frac{1}{b}\right)^a\right) > b^a\left(a - \left(1+\frac{1}{b}\right)^b\right) > b^a(a-e) > 0$$

and

$$a^2 + ab - a - b - 1 = a^2 + b(a-1) - a - 1 \geqslant a^2 + (a^2-1) - a - 1 > 0.$$

Also solved by the problem proposer.

• **5760** *Proposed by Michel Bataille, Rouen, France.*

Let a, b be real numbers such that $0 < a < b$. Prove that

$$(a+b)^{a+b}(b-a)^{b-a} > (a^2+b^2)^b.$$

Solution 1 by proposed by Moti Levy, Rehovot, Israel.

Let

$$x := a+b, \quad y := b-a,$$

then the inequality in terms of x and y is

$$\frac{x\ln\left(x^2\right) + y\ln\left(y^2\right)}{2} > \frac{x+y}{2}\ln\left(\frac{x^2+y^2}{2}\right), \quad x>0, \quad y>0.$$

The function

$$f(t) := t\ln\left(t^2\right)$$

is convex for $t > 0$ since $f''(t) = \dfrac{2}{t} > 0$ for $t > 0$. Hence by Jensen's inequality the original inequality is true.

Solution 2 by proposed by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata," Roma, Italy.

$$(a+b)^{a+b}(b-a)^{b-a} > (a^2+b^2)^b = b^{a+b}\left(\frac{a}{b}+1\right)^{b(1-\frac{b}{a})} b^{b-a}\left(1-\frac{a}{b}\right)^{b(1-\frac{a}{b})} > b^{2b}\left(1+\frac{a^2}{b^2}\right).$$

If $a/b = x \in (0,1)$ the inequality becomes

$$(1-x)^{1-x}(1+x)^{1+x} > 1+x^2 \iff \left(\frac{1+x}{1-x}\right)^x > \frac{1+x^2}{1-x^2}$$

which in turn it is equivalent to

$$e^{x(\ln(1+x)-\ln(1-x))} > e^{\ln(1+x^2)-\ln(1-x^2)} \iff x(\ln(1+x)-\ln(1-x) > \ln(1+x^2)-\ln(1-x^2)$$

$$\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}, \quad \ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}, \quad 0 \leqslant x < 1$$

$$\ln(1+x)-\ln(1-x) = \sum_{k=0}^{\infty}\frac{2x^{2k+1}}{2k+1}$$

hence $x(\ln(1+x)-\ln(1-x) > \ln(1+x^2)-\ln(1-x^2)$ is equivalent to

$$\sum_{k=0}^{\infty}\frac{2x^{2k+1}}{2k+1} > \sum_{k=0}^{\infty}\frac{2x^{4k+2}}{2k+1}$$

which evidently holds true by $0 \leqslant x < 1$.

Also solved by Albert Stadler, Herrliberg, Switzerland; and the problem proposer.

• **5761** *Proposed by Narendra Bhandari and Yogesh Joshi, Nepal.*

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{\left[\frac{n}{2}\right]}-H_{\left[\frac{n-1}{2}\right]}\right)}{4^n(6n+3)} + \int_0^{\frac{\pi}{4}}\frac{4y\sec y dy}{\sqrt{9\cos 2y}} = \zeta(2)$$

where $H_{[n]} = \displaystyle\int_0^1 \frac{1-x^n}{1-x}dx$ and $\zeta(n) = \displaystyle\sum_{k=1}^{\infty}\frac{1}{k^n}$ is Riemann zeta function for $n > 1$.

Solution 1 by proposed by Albert Stadler, Herrliberg, Switzerland.

We have

$$\sum_{n=0}^{\infty}\begin{pmatrix}2n\\n\end{pmatrix}\frac{\left(H_{\left[\frac{n}{2}\right]}-H_{\left[\frac{n-1}{2}\right]}\right)}{4^n(6n+3)} = \frac{1}{3}\sum_{n=0}^{\infty}\begin{pmatrix}2n\\n\end{pmatrix}\frac{1}{4^n(2n+1)}\int_0^1\frac{x^{\frac{n-1}{2}}-x^{\frac{n}{2}}}{1-x}dx =$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^n(2n+1)}\int_0^1 \frac{x^{\frac{n-1}{2}}}{1+\sqrt{x}}dx \overbrace{=}^{x=y^4} \frac{4}{3}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^n(2n+1)}\int_0^1 \frac{y^{2n+1}}{1+y^2}dy =$$

$$= \frac{4}{3}\int_0^1 \frac{\arcsin y}{1+y^2}dy \overbrace{=}^{y=\tan z} \frac{4}{3}\int_0^{\frac{\pi}{4}} \arcsin(\tan z)\, dz,$$

where we have used Taylor's expansion of arcsin(.), namely

$$\arcsin x = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{2n+1}}{4^n(2n+1)}.$$

The interchange of summation and integration is permitted, since all involved terms are positive. We note that

$$\int \frac{\sec y}{\sqrt{\cos(2y)}}dy = \int \frac{1}{\cos y\sqrt{\cos^2 y - \sin^2 y}}dy = \int \frac{1}{\cos^2 y\sqrt{1-\tan^2 y}}dy = \arcsin(\tan y) + C.$$

Integration by parts then gives

$$\int_0^{\frac{\pi}{4}} \frac{4y\sec y\, dy}{\sqrt{9\cos(2y)}} = \frac{4y}{3}\arcsin(\tan y)\Big|_0^{\frac{\pi}{4}} - \frac{4}{3}\int_0^{\frac{\pi}{4}} \arcsin(\tan y)\, dy =$$

$$= \frac{\pi^2}{6} - \frac{4}{3}\int_0^{\frac{\pi}{4}} \arcsin(\tan y)\, dy = (2) - \frac{4}{3}\int_0^{\frac{\pi}{4}} \arcsin(\tan y)\, dy.$$

Hence $\displaystyle\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{\left[\frac{n}{2}\right]} - H_{\left[\frac{n-1}{2}\right]}\right)}{4^n(6n+3)} + \int_0^{\frac{\pi}{4}} \frac{4y\sec y\, dy}{\sqrt{9\cos(2y)}} = (2).$

Solution 2 by proposed by Moti Levy, Rehovot, Israel.

The equality in the problem statement can be rewritten as

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = 3\zeta(2) - 4\int_0^{\frac{\pi}{4}} \frac{y\sec(y)}{\sqrt{\cos(2y)}}dy \tag{4}$$

By change of integration variable and the fact $\zeta(2) = \frac{\pi^2}{6}$, (4) is equivalent to

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = \frac{\pi^2}{2} - 4\int_0^1 \frac{\arctan(t)}{\sqrt{1-t^2}}dt. \tag{5}$$

By definition of $H_x := \int_0^1 \frac{1-t^x}{1-t}dx$,

$$H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = \int_0^1 \frac{1-t^{\frac{n}{2}}}{1-t} - \frac{1-t^{\frac{n}{2}-\frac{1}{2}}}{1-t}dt$$

$$= \int_0^1 t^{\frac{n-1}{2}}\frac{1-\sqrt{t}}{1-t}dt \tag{6}$$

Plugging (6) into the left hand side of (5) and then exchanging the order of summation and integration, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right)\\
&=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\int_0^1 t^{\frac{n-1}{2}}\frac{1-\sqrt{t}}{1-t}dt\\
&=\int_0^1\frac{1-\sqrt{t}}{\sqrt{t}(1-t)}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\left(\sqrt{t}\right)^n dt. \qquad (7)
\end{aligned}$$

The power series expansion of $\arcsin(z)$ is

$$\arcsin(z)=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}z^{2n+1} \qquad (8)$$

Applying (8) to (7) we get

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right)=\int_0^1\frac{1-\sqrt{t}}{\sqrt{t}(1-t)}\frac{\arcsin\left(\sqrt[4]{t}\right)}{\sqrt[4]{t}}dt.$$

Changing the integration variable $\sqrt[4]{t}=x$,

$$\int_0^1\frac{1-\sqrt{t}}{\sqrt{t}(1-t)}\frac{\arcsin\left(\sqrt[4]{t}\right)}{\sqrt[4]{t}}dt=4\int_0^1\frac{\arcsin(x)}{x^2+1}dx$$

and then by integration by parts,

$$4\int_0^1\frac{\arcsin(x)}{x^2+1}dx=\frac{1}{2}\pi^2-4\int_0^1\frac{\arctan(x)}{\sqrt{1-x^2}}dx.$$

Thus equation (5) is proved.

Solution 3 by proposed by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\because \quad H_{\left[\frac{n}{2}\right]}-H_{\left[\frac{n-1}{2}\right]}=2\int_0^1\frac{x^n}{1+x}\mathrm{d}x$$

$$\therefore \quad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{\left[\frac{n}{2}\right]}-H_{\left[\frac{n-1}{2}\right]}\right)}{4^n(6n+3)}=\frac{2}{3}\int_0^1\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{(1+x)4^n(2n+1)}\mathrm{d}x$$

$$\because \quad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{4^n(2n+1)}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\sqrt{x}^{2n}}{4^n(2n+1)}$$

$$=\frac{1}{\sqrt{x}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\sqrt{x}^{2n+1}}{4^n(2n+1)}=\frac{1}{\sqrt{x}}\arcsin(\sqrt{x})$$

$$\therefore \quad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{\left[\frac{n}{2}\right]}-H_{\left[\frac{n-1}{2}\right]}\right)}{4^n(6n+3)} = \frac{2}{3}\int_0^1 \frac{\arcsin(\sqrt{x})}{(1+x)\sqrt{x}}\mathrm{d}x$$

$$= \frac{2}{3}\int_0^1 \frac{\arcsin y}{(1+y^2)y}2y\mathrm{d}y \quad (\text{let, } y = \sqrt{x})$$

$$= \frac{4}{3}\int_0^1 \frac{\arcsin y}{1+y^2}\mathrm{d}y$$

Now it's equal to prove

$$\int_0^{\frac{\pi}{4}} \frac{y}{\cos y\sqrt{\cos(2y)}}\mathrm{d}y + \int_0^1 \frac{\arcsin y}{1+y^2}\mathrm{d}y = \frac{3}{4}\zeta(2)$$

$$\Leftrightarrow \quad \int_0^1 \frac{\arctan x}{\sqrt{1-x^2}}\mathrm{d}x + \int_0^1 \frac{\arcsin x}{1+x^2}\mathrm{d}x = \frac{3}{4}\zeta(2)$$

$$\because \quad \int\left(\frac{\arctan x}{\sqrt{1-x^2}} + \frac{\arcsin x}{1+x^2}\right)\mathrm{d}x = \arcsin x \arctan x + C$$

$$\therefore \quad LHS = \arcsin x \arctan x\Big|_0^1 = \arcsin(1)\arctan(1) = \frac{\pi}{4}\times\frac{\pi}{2}$$

$$= \frac{3}{4}\times\frac{\pi^2}{6} = \frac{3}{4}\zeta(2) = RHS.$$

Also solved by Narendra Bhandari and Yogesh Joshi, Nepal; and the problem proposer.

• **5762** *Proposed by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Rome, Italy.*

Let $f\colon [-1,1] \to \mathbb{R}$ be a three-times continuously differentiable such that $f(-1) = f'(-1) = f''(1) = 0$. Prove that

$$\left(\int_{-1}^1 f(x)dx\right)^2 \leqslant \frac{34}{315}\int_{-1}^1 \left(f'''(x)\right)^2 dx.$$

Solution 1 by proposed by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

The indicated inequality is incorrect. As a counterexample, consider the function $f(x) = x^3 - 3x^2 - 9x - 5$. Then $f(-1) = f'(-1) = f''(1) = 0$, and

$$\int_{-1}^1 f(x)\,dx = -12 \quad \text{and} \quad \int_{-1}^1 (f'''(x))^2\,dx = 72.$$

Thus,

$$\left(\int_{-1}^1 f(x)\,dx\right)^2 = 144 > \frac{272}{35} = \frac{34}{315}\int_{-1}^1 (f'''(x))^2\,dx.$$

To establish the correct inequality, we proceed as follows. Integrate by parts with $u = f(x)$ and $dv = dx$. Because $f(-1) = 0$, we choose as v the antiderivative which evaluates to 0 at $x = 1$; that is, choose $v = x - 1$. Then

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= (x-1)f(x)\Big|_{-1}^{1} - \int_{-1}^{1}(x-1)f'(x)\,dx \\ &= -\int_{-1}^{1}(x-1)f'(x)\,dx.\end{aligned}$$

Next, integrate by parts with $u = f'(x)$, $dv = (x-1)\,dx$, and again choose as v the antiderivative which evaluates to 0 at $x = 1$; that is, choose $v = \frac{1}{2}(x-1)^2$. This yields

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= -\frac{1}{2}(x-1)^2 f'(x)\Big|_{-1}^{1} + \int_{-1}^{1}\frac{1}{2}(x-1)^2 f''(x)\,dx \\ &= \int_{-1}^{1}\frac{1}{2}(x-1)^2 f''(x)\,dx.\end{aligned}$$

One more integration by parts with $u = f''(x)$, $dv = \frac{1}{2}(x-1)^2$, and $v = \frac{1}{6}(x-1)^3 + \frac{4}{3}$ – the antiderivative which evaluates to 0 at $x = 1$ – gives

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= \left(\frac{1}{6}(x-1)^3 + \frac{4}{3}\right) f''(x)\Big|_{-1}^{1} - \int_{-1}^{1}\left(\frac{1}{6}(x-1)^3 + \frac{4}{3}\right) f'''(x)\,dx \\ &= -\int_{-1}^{1}\left(\frac{1}{6}(x-1)^3 + \frac{4}{3}\right) f'''(x)\,dx.\end{aligned}$$

Now, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\int_{-1}^{1} f(x)\,dx\right)^2 &= \left(\int_{-1}^{1}\left(\frac{1}{6}(x-1)^3 + \frac{4}{3}\right) f'''(x)\,dx\right)^2 \\ &\leqslant \int_{-1}^{1}\left(\frac{1}{6}(x-1)^3 + \frac{4}{3}\right)^2 dx \cdot \int_{-1}^{1}(f'''(x))^2\,dx \\ &= \frac{16}{7}\int_{-1}^{1}(f'''(x))^2\,dx.\end{aligned}$$

Equality occurs if and only if $f'''(x)$ is a scalar multiple of $\frac{1}{6}(x-1)^3 + \frac{4}{3}$; taking into account the conditions $f(-1) = f'(-1) = f''(1) = 0$, it follows that equality holds if and only if $f(x)$ is a scalar multiple of $(x+1)^2(x^4 - 8x^3 + 30x^2 + 88x - 671)$.

Solution 2 by proposed by Michel Bataille, Rouen, France.

The proposed inequality cannot hold for all the functions satisfying the given hypotheses: consider the polynomial function p defined by $p(x) = x^3 - 3x^2 - 9x - 5$. It is readily checked that $p(-1) = p'(-1) = p''(1) = 0$. We easily obtain that $p'''(x) = 6$ so that $\int_{-1}^{1} (p'''(x))^2\,dx = 72$. Since

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} (-3x^2 - 5)\,dx = 2\int_{0}^{1} (-3x^2 - 5)\,dx = -12$$

we have

$$\left(\int_{-1}^{1} p(x)\,dx\right)^2 = 2\int_{-1}^{1} (p'''(x))^2\,dx,$$

contradicting the proposed inequality.
However, under the given hypotheses, we can prove the following inequality:

$$\left(\int_{-1}^{1} f(x)\,dx\right)^2 \leqslant \frac{16}{7}\int_{-1}^{1} (f'''(x))^2\,dx.$$

Indeed, let $q(x) = x^3 - 3x^2 + 3x + 7$. Then $q(-1) = q'(1) = q''(1) = 0,\ q'''x) = 6$ and

$$\begin{aligned}\int_{-1}^{1} f'''(x)q(x)\,dx &= [f''(x)q(x)]_{-1}^{1} - \int_{-1}^{1} q'(x)f''(x)\,dx \\ &= -[q'(x))f'(x)]_{-1}^{1} + \int_{-1}^{1} q''(x)f'(x)\,dx \\ &= [q''(x)f(x)]_{-1}^{1} - 6\int_{-1}^{1} f(x)\,dx = -6\int_{-1}^{1} f(x)\,dx.\end{aligned}$$

Therefore, from the Cauchy-Schwarz inequality we have

$$\left(-6\int_{-1}^{1} f(x)\,dx\right)^2 = \left(\int_{-1}^{1} f'''(x)q(x)\,dx\right)^2 \leqslant \left(\int_{-1}^{1} (f'''(x))^2\,dx\right)\left(\int_{-1}^{1} (q(x))^2\,dx\right),$$

from which we easily get the desired inequality (since $\int_{-1}^{1} (q(x))^2\,dx = \frac{576}{7}$).

Solution 3 by proposed by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\begin{aligned}&\int_{-1}^{1} f'''(x)g(x)\mathrm{d}x = \int_{-1}^{1} g(x)\mathrm{d}f''(x) \\ &\quad = g(x)f''(x)\Big|_{-1}^{1} - \int_{-1}^{1} f''(x)\mathrm{d}g(x) \\ &\quad = g(x)f''(x)\Big|_{-1}^{1} - \int_{-1}^{1} f''(x)g'(x)\mathrm{d}x\end{aligned}$$

$$= -g(-1)f''(-1) - \int_{-1}^{1} g'(x)\mathrm{d}f'(x)$$

$$= -g(-1)f''(-1) - g'(x)f'(x)\Big|_{-1}^{1} + \int_{-1}^{1} f'(x)\mathrm{d}g'(x)$$

$$= -g(-1)f''(-1) - g'(x)f'(x)\Big|_{-1}^{1} + \int_{-1}^{1} g''(x)\mathrm{d}f(x)$$

$$= -g(-1)f''(-1) - g'(1)f'(1) + g''(x)f(x)\Big|_{-1}^{1} - \int_{-1}^{1} f(x)\mathrm{d}g''(x)$$

$$= -g(-1)f''(-1) - g'(1)f'(1) + g''(1)f(1) - \int_{-1}^{1} f(x)g'''(x)\mathrm{d}x$$

Let, $g(x) = ax^3 + bx^2 + cx + d$

then, $g'(x) = 3ax^2 + 2bx + c, \quad g''(x) = 6ax + 2b, \quad g'''(x) = 6a$

Let, $g(-1) = 0, \quad g'(1) = 0, \quad g''(1) = 0$

then, $-a + b - c + d = 0, \quad 3a + 2b + c = 0, \quad 6a + 2b = 0,$

$\Rightarrow \quad b = -3a, \quad c = 3a, \quad d = 7a, \quad g(x) = a(x^3 - 3x^2 + 3x + 7)$

and $\int_{-1}^{1} f'''(x)g(x)\mathrm{d}x = -\int_{-1}^{1} f(x)g'''(x)\mathrm{d}x$

According to Cauchy-Buniakowsky-Schwarz Inequality: $\left(\int f(x)g(x)\mathrm{d}x\right)^2 \leqslant \int f^2(x)\mathrm{d}x \int g^2(x)\mathrm{d}x$

$$\left(\int_{-1}^{1} f'''(x)g(x)\mathrm{d}x\right)^2 = \left(-\int_{-1}^{1} f(x)g'''(x)\mathrm{d}x\right)^2 \leqslant \int_{-1}^{1} \left(f'''(x)\right)^2 \mathrm{d}x \int_{-1}^{1} \left(g(x)\right)^2 \mathrm{d}x$$

$$\Leftrightarrow \quad \left(\int_{-1}^{1} f(x)\mathrm{d}x\right)^2 \times 36a^2 \leqslant \int_{-1}^{1} \left(f'''(x)\right)^2 \mathrm{d}x \int_{-1}^{1} a^2 \left(x^3 - 3x^2 + 3x + 7\right)^2 \mathrm{d}x$$

$$\Leftrightarrow \quad \left(\int_{-1}^{1} f(x)\mathrm{d}x\right)^2 \leqslant \frac{1}{36}\int_{-1}^{1} \left(f'''(x)\right)^2 \mathrm{d}x \int_{-1}^{1} \left(x^3 - 3x^2 + 3x + 7\right)^2 \mathrm{d}x$$

$$\Leftrightarrow \quad \left(\int_{-1}^{1} f(x)\mathrm{d}x\right)^2 \leqslant \frac{1}{36}\int_{-1}^{1} \left(f'''(x)\right)^2 \mathrm{d}x \times \frac{576}{7}$$

$$\Leftrightarrow \quad \left(\int_{-1}^{1} f(x)\mathrm{d}x\right)^2 \leqslant \frac{16}{7}\int_{-1}^{1} \left(f'''(x)\right)^2 \mathrm{d}x.$$

It's $\frac{16}{7}$, not $\frac{34}{315}$.

Also solved by the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

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3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣