Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before September 15, 2017

• **5451**: Proposed by Kenneth Korbin, New York, NY

Given triangle $ABC$ with sides $a = 8, b = 19$ and $c = 22$. The triangle has an interior point $P$ where $AP$, $BP$, and $CP$ each have positive integer length. Find $AP$ and $BP$, if $CP = 4$.

• **5452**: Proposed by Roger Izard, Dallas, TX

Let point $O$ be the orthocenter of a given triangle $ABC$. In triangle $ABC$ let the altitude from $B$ intersect line segment $AC$ at $E$, and the altitude from $C$ intersect line segment $AB$ at $D$. If $AC$ and $AB$ are unequal, derive a formula which gives the square of $BC$ in terms of $AC, AB, EO$, and $OD$.

• **5453**: Proposed by D.M. Bătinețu-Giurgiu, “Matei Basarab” National College, Bucharest, Romania and Neculai Stanciu, “George Emil Palade” School, Buzău, Romania

If $a, b, c \in (0, 1)$ or $a, b, c \in (1, \infty)$ and $m, n$ are positive real numbers, then prove that

$$\frac{\log_a b + \log_a c}{m + n \log_a a} + \frac{\log_b c + \log_b a}{m + n \log_b b} + \frac{\log_c a + \log_c b}{m + n \log_c c} \geq \frac{6}{m + n}$$

• **5454**: Proposed by Arkady Alt, San Jose, CA

Prove that for integers $k$ and $l$, and for any $\alpha, \beta \in (0, \frac{\pi}{2})$, the following inequality holds:

$$k^2 \tan \alpha + l^2 \tan \beta \geq \frac{2kl}{\sin(\alpha + \beta)} - (k^2 + l^2) \cot(\alpha + \beta).$$

• **5455**: Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Find all real solutions to the following system of equations:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc}$$

$$a + b + c = abc + \frac{8}{27} (a + b + c)^3$$
Let \( k \) be a positive integer. Calculate
\[
\lim_{x \to \infty} e^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left( e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right).
\]

### Solutions

**5433: Proposed by Kenneth Korbin, New York, NY**

Solve the equation: \( \sqrt[4]{x} + x^2 = \sqrt[4]{x} + \sqrt[4]{x - x^2}, \) with \( x > 0. \)

**Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND**

Let \( f(x) = \sqrt[4]{x} + \sqrt[4]{x - x^2} - \sqrt[4]{x + x^2}. \) Then \( f(x) \) is continuous on \([0,1]\). We have \( f(1/2) > 0 \) and \( f(1) < 0. \) By the Intermediate Value Theorem our original equation has at least one solution with \( x > 0. \)

Now consider
\[
\sqrt[4]{x + x^2} = \sqrt[4]{x} + \sqrt[4]{x - x^2} \implies \sqrt[4]{1 + x} = 1 + \sqrt[4]{1 - x}
\]
\[
\implies \sqrt[4]{1 + x} - \sqrt[4]{1 - x} = 1 \implies \sqrt[4]{1 + x} - 2 \sqrt[4]{1 - x^2} + \sqrt[4]{1 - x} = 1 \implies \sqrt[4]{1 + x} + \sqrt[4]{1 - x} = 1 + 2 \sqrt[4]{1 - x^2} \implies 1 + x + 2 \sqrt[4]{1 - x^2} + 1 - x = 1 + 4 \sqrt[4]{1 - x^2} + 4 \sqrt[4]{1 - x^2} \implies 1 - 2 \sqrt[4]{1 - x^2} = 4 \sqrt[4]{1 - x^2} \implies 1 - 4 \sqrt[4]{1 - x^2} + 4(1 - x^2) = 16 \sqrt[4]{1 - x^2} \implies 5 - 4x^2 = 20(1 - x^2) \implies 25 - 40x^2 + 16x^4 = 400(1 - x^2) \implies 16x^4 + 360x^2 - 375 = 0
\]

As a quadratic in \( x^2 \) the roots of this polynomial are
\[
x^2 = \frac{-360 \pm 160\sqrt{6}}{32} = -\frac{45 \pm 20\sqrt{6}}{4}
\]

and so
\[
x = \pm \frac{\sqrt{-45 \pm 20\sqrt{6}}}{2}
\]

This is a positive real number only if we choose both signs positive. Thus our original equation has at most one positive real solution.

Our last two paragraphs show that
\[
x = \frac{\sqrt{20\sqrt{6} - 45}}{2}.
\]
is the unique positive real solution to our original equation.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since \( x > 0 \), we lose no solutions if we divide by \( \sqrt[4]{x} \) to obtain

\[
\sqrt{1+x} = 1 + \sqrt{1-x}.
\]

If we let \( X = \sqrt{1+x} \) and \( Y = \sqrt{1-x} \), then \( X^4 + Y^4 = 2 \) and we can solve for \( XY \) in the following steps:

\[
\begin{align*}
X - Y &= 1 \\
(X - Y)^4 &= 1 \\
X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4 &= 1 \\
X^4 + Y^4 - 2XY(2X^2 - 3XY + 2Y^2) &= 1 \\
-2XY[2(X - Y)^2 + XY] &= -1 \\
2XY(XY + 2) &= 1 \\
2X^2Y^2 + 4XY - 1 &= 0 \\
XY &= \frac{-2 \pm \sqrt{6}}{2}.
\end{align*}
\]

The condition \( XY = \sqrt{1-x^2} \geq 0 \) implies that

\[
\sqrt{1-x^2} = \frac{\sqrt{6} - 2}{2}
\]

\[
1 - x^2 = \left( \frac{\sqrt{6} - 2}{2} \right)^4 = \frac{49 - 20\sqrt{6}}{4}
\]

\[
x^2 = 1 - \frac{49 - 20\sqrt{6}}{4} = \frac{20\sqrt{6} - 45}{4}.
\]

Because \( x > 0 \), our solution is

\[
x = \frac{\sqrt{20\sqrt{6} - 45}}{2}.
\]

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Solution. Since \( x > 0 \), we may divide the given equation by \( \sqrt{x} \) to produce

\[
\sqrt{1+x} = 1 + \sqrt{1-x}.
\]

Squaring both sides then yields \( \sqrt{1+x} = 1 + 2\sqrt{1-x} + \sqrt{1-x} \), or \( \sqrt{1+x} - \sqrt{1-x} - 1 = 2\sqrt{1-x} \). Squaring yet again produces

\[
(1+x) + (1-x) + 1 - 2\sqrt{1+x} + 2\sqrt{1-x} - 2\sqrt{1-x^2} = 4\sqrt{1-x},
\]

or \( 3 - 2\sqrt{1-x^2} = 2\sqrt{1+x} + 2\sqrt{1-x} \). We square once more to obtain

\[
9 - 12\sqrt{1-x^2} + 4(1-x^2) = 4(1+x) + 4(1-x) + 8\sqrt{1-x^2}
\]
and thus $5 - 4x^2 = 20\sqrt{1 - x^2}$. Squaring for the last time yields
$25 - 40x^2 + 16x^4 = 400(1 - x^2)$ and hence $16x^4 + 360x^2 - 375 = 0$. Finally, the only real positive solution of this equation is
$$x = \sqrt{-\frac{45}{4} + 5\sqrt{6}} = \frac{\sqrt{-45 + 20\sqrt{6}}}{2}.$$ 

Addendum. It is interesting to note that this solution is approximately 0.99872354, very close to 1. In particular, this implies that $49/4$ is a good rational approximation of $5\sqrt{6}$, which also means that $7/2$ is a good rational approximation of $\sqrt{150}$.

Also solved by Arkady Alt, San Jose, CA; Hafiz I. Arshagi, Guilford Technical Community College, Jamestown, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Aykut Ismailov, Shumen, Bulgaria; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY at Oneonta, Oneonta, NY; Boris Rays, Brooklyn, NY; Brandon Richardson (student), Auburn University at Montgomery, AL; Toshihiro Shimizu, Kawasaki, Japan; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

5434: Proposed by Titu Zvonaru, Comnesti, Romania and Neculai Stanciu, “George Emil Palade” General School, Buzău, Romania

Calculate, without using a calculator or log tables, the number of digits in the base 10 expansion of $2^{96}$.

Solution 1 by Ed Gray, Highland Beach, FL

$$(2^{12})^8 = 2^{96} > (4 \cdot 10^3)^8 = 4^8 \cdot 10^{24} > 6 \cdot 10^4 \cdot 10^{24} = 6 \cdot 10^{28}.$$ 

Also
$$(2^8)^{12} = 2^{96} < (3 \cdot 10^2)^{12} = 3^{12} \cdot 10^{24} < (6 \cdot 10^5) \cdot 10^{24} = 6 \cdot 10^{29}.$$ 

Therefore, $6 \cdot 10^{28} < 2^{96} < 6 \cdot 10^{29}$. So $n = 29$.

Solution 2 by Paul M. Harms, North Newton, KS

We see that
$$4(10^3) < 2^{12} = 4096 < 4.1(10^4).$$ 

Then
$$16(10^6) < 2^{24} < 16.81(10^6) < 17(10^6).$$ 

Taking the fourth power of the appropriate terms we obtain,
$$16^4(10^{24}) = 65536(10^{24}) = 0.65536(10^{29}) < 2^{96} < 17^4(10^{24}) = 83521(10^{24}) = 0.83521(10^{29}).$$ 

Since $2^{96}$ is bounded by integers who have 29 digits in the base 10 expansion, the integer $2^{96}$ must also have 29 digits in its base 10 expansion.
Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

The required number of digits is 29 because, as we shall show, \(10^{28} \leq 2^{96} < 10^{29}\). More exactly, we shall prove that \(1 < \frac{2^{96}}{10^{28}} < 10\). Since

\[
\frac{2^{96}}{10^{28}} = \left(\frac{2^{24}}{10^7}\right)^4 = \left(\frac{2^{12}}{10^7}\right)^4 = \left(\frac{4096^2}{10^7}\right)^4 = \left(\frac{1,6777216 \cdot 10^7}{10^7}\right)^4 = (1,6777216)^4,
\]

we obtain that

\[1^4 < \frac{2^{96}}{10^{28}} < 1.68^4,\]

that is \(1 < \frac{2^{96}}{10^{28}} < (2.8224)^2\) and, hence, \(1 < \frac{2^{96}}{10^{28}} < 3^2 < 10\).

Note: another way to show that \(10^{28} < 2^{96}\) is, for example:

\[
5^2 < 2^5 \quad \Rightarrow \quad 5^3 < 2^9 \quad \Rightarrow \quad 5^5 < 2^{12} \quad \Rightarrow \quad 5^7 < 2^{17} \quad \Rightarrow \quad 2^7 \cdot 5^7 < 2^{24} \quad \Rightarrow \quad (10^7)^4 < (2^{24})^4 \quad \Rightarrow \quad 10^{28} < 2^{96}.
\]

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

Since \(10^3 < 2^{10} = 1024 < 1.03 \times 10^3\) and \(2^{96} = (2^{10})^9 \times 2^6 = (2^{10})^9 \times 10 \times 6.4\) we have

\[6.4 \times 10 \times 10^3 \times 9 < 2^{96} < 6.4 \times 10 \times 10^3 \times 9 \times (1.03)^9.\]

We evaluate 1.03\(^9\). We have \(1.03 \times 1.03 \times 1.03 = 1.0609 \times 1.03 = 1.092727 < 1.1\) and \(1.1 \times 1.1 \times 1.1 = 1.331 < 1.4\) (I never use calculator.) Therefore, we have

\[10^{28} < 6.4 \times 10^{28} < 2^{96} < 6.4 \times 1.4 \times 10^{28} = 8.96 \times 10^{28} < 10^{29}.
\]

Therefore, the number of digits in \(2^{96}\) is 29.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Hatem I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

• 5435: Proposed by Valcho Milchev, Petko Rachov Slaveikov Secondary School, Bulgaria

Find all positive integers \(a\) and \(b\) for which \(\frac{a^4 + 3a^2 + 1}{ab - 1}\) is a positive integer.

Solution 1 by Moti Levy, Rehovot, Israel

This solution is based on similar problem and solution which appeared in [1].
\[
\frac{a^4 + 3a^2 + 1}{ab - 1}
\] may be replaced by equivalent expression with symmetric polynomial in the numerator.

Indeed,
\[
\frac{a^4 + 3a^2 + 1}{ab - 1} = a^2 (a^2 + b^2 + 3) - (ab - 1)(ab + 1).
\]

Now, \(a\) and \(ab - 1\) satisfy the equation \(b \cdot a + (-1) \cdot (ab - 1) = 1\), which implies that \(a\)
and \(ab - 1\) are relatively prime and clearly \(a^2\) and \(ab - 1\) are also relatively prime.
Thus, \(\frac{a^4 + 3a^2 + 1}{ab - 1}\) is a positive integer if and only if \(\frac{a^2 + b^2 + 3}{ab - 1}\) is a positive integer.

We call the ordered pair \((a, b)\) a solution if
\[
\frac{a^2 + b^2 + 3}{ab - 1} = m,
\]
where \(m\) is a positive integer. The set of solutions is not empty since \((1, 2)\) is a solution.

We exclude \((a, a)\) from the set of solutions since \(2a^2 + 3\) is not a positive integer for all \(a > 0\).
Equation (1) is re-written as follows
\[
a^2 - mab + b^2 = -(m + 3).
\]

It is easily verified (see (3)) that if \((a, b)\) is a solution then \((ma - b, a)\) is a solution as well.

\[
(ma - b)^2 - m (ma - b) a + a^2 = a^2 - mab + b^2,
\]

Let \((a_0, b_0)\) be the “smallest” solution in the sense that \(a_0 + b_0 \leq a + b\), where \((a, b)\) is any solution.

\[
a_0 + b_0 \leq (ma_0 - b_0) + a_0,
\]
or
\[
\frac{2b_0}{a_0} \leq m.
\]

\[
\frac{2b_0}{a_0} \leq \frac{a_0^2 + b_0^2 + 3}{a_0b_0 - 1}
\]
\[
0 \leq -2a_0b_0^2 + 2b_0 + a_0^3 + 3a_0
\]

Let \((a_0, a_0 + k)\) be a solution. Then substituting in (5) gives,
\[
0 \leq -2a_0 (a_0 + k)^2 + 2 (a_0 + k) + a_0^3 + 3a_0
\]
\[
= -2k^2a_0 - 4ka_0^2 + 2k - a_0^3 + 5a_0.
\]

Solving \(-2k^2a_0 - 4ka_0^2 + 2k - a_0^3 + 5a_0 \geq 0\), we get
\[
\frac{1}{2a_0} \left(1 - 2a_0^2 - \sqrt{6a_0^2 + 2a_0^4 + 1}\right) \leq k \leq \frac{1}{2a_0} \left(1 - 2a_0^2 + \sqrt{6a_0^2 + 2a_0^4 + 1}\right),
\]
hence, \(k\) will have positive values only if
\[
\sqrt{6a_0^2 + 2a_0^4 + 1} + 1 \geq 2a_0.
\]

This inequality holds for \(a_0 = 1\) and \(a_0 = 2\). For \(a_0 = 1\), possible values for \(k\) are 1 or 2; for \(a_0 = 2\), possible value for \(k\) is 1.
Thus we have to check the following set of potential solutions: \{\( (1, 2), (1, 3), (2, 1) \) \}. Clearly (1, 2) and (2, 1) are solutions, but (1, 3) is not.

For (1, 2) and (2, 1) the value of \( m \) is 8. We conclude that the sole value of \( m \) is 8.

It follows from (3) that the pairs \((a_n, b_n)\) (and by symmetry \((b_n, a_n)\)), which satisfy condition (1) are expressed by the recurrence formulas

\[
\begin{align*}
a_{n+1} &= 8a_n - b_n, \\
b_{n+1} &= a_n,
\end{align*}
\]

which are equivalent to the recurrence formulas

\[
\begin{align*}
a_{n+2} &= 8a_{n+1} - a_n, \\
b_{n+2} &= 8b_{n+1} - b_n.
\end{align*}
\]

We have two sets of initial conditions:

1) \( a_0 = 1, a_1 = 6, b_0 = 2, b_1 = 1 \); the pairs resulting from these initial conditions are \((1, 2), (6, 1), (47, 6), (370, 47), \ldots \).

\[
\begin{align*}
a_n &= \left( \frac{1}{2} - \frac{1}{\sqrt{15}} \right) \left( 4 - \sqrt{15} \right)^n + \left( \frac{1}{2} + \frac{1}{\sqrt{15}} \right) \left( 4 + \sqrt{15} \right)^n, \\
b_n &= \left( 1 + \frac{7}{2\sqrt{15}} \right) \left( 4 - \sqrt{15} \right)^n + \left( 1 - \frac{7}{2\sqrt{15}} \right) \left( 4 + \sqrt{15} \right)^n.
\end{align*}
\]

2) \( a_0 = 2, a_1 = 15, b_0 = 1, b_1 = 2 \); the pairs resulting from these initial conditions are \((2, 1), (15, 2), (118, 15), (929, 118), \ldots \).

\[
\begin{align*}
a_n &= \left( 1 - \frac{7}{2\sqrt{15}} \right) \left( 4 - \sqrt{15} \right)^n + \left( 1 + \frac{7}{2\sqrt{15}} \right) \left( 4 + \sqrt{15} \right)^n, \\
b_n &= \left( \frac{1}{2} + \frac{1}{\sqrt{15}} \right) \left( 4 - \sqrt{15} \right)^n + \left( \frac{1}{2} - \frac{1}{\sqrt{15}} \right) \left( 4 + \sqrt{15} \right)^n.
\end{align*}
\]

Reference:


Solution 2 by Anthony Bevelacqua, University of North Dakota, Grand Forks, ND

1) There are no solutions to our problem with \( a = b \). We have \( a^4 + 3a^2 + 1 \equiv 5 \pmod{(a^2 - 1)} \). Assume there is a solution with \( a = b \). Then \( a^2 - 1 \) divides \( a^4 + 3a^2 + 1 \) so \( a^4 + 3a^2 + 1 \equiv 0 \pmod{(a^2 - 1)} \). Thus \( 5 \equiv 0 \pmod{(a^2 - 1)} \) and so \( a^2 - 1 \) divides 5. But then \( a^2 = 2 \) or \( a^2 = 6 \), a contradiction in either case.

2) The only solutions with \( a \leq 4 \) are \( (a, b) = (1, 2), (2, 1), (1, 6) \) and \( (2, 15) \). Suppose \((a, b)\) is a solution to our problem. If \( a = 1 \) then \( b - 1 \) divides 5 so \( b - 1 = 1 \) or \( b - 1 = 5 \). Both \((1, 2)\) and \((1, 6)\) are solutions. If \( a = 2 \) then \( 2b - 1 \) divides 29 so \( 2b - 1 = 1 \) or \( 2b - 1 = 29 \). Both \((2, 1)\) and \((2, 15)\) are solutions. If \( a = 3 \) then \( 3b - 1 \) divides 109 so \( 3b - 1 = 1 \) or \( 3b - 1 = 109 \), a contradiction. If \( a = 4 \) then \( 4b - 1 \) divides 305 = 5 \cdot 61 \) so \( 4b - 1 \in \{1, 5, 61, 305\} \), a contradiction.
3) \(ab - 1\) divides \(a^4 + 3a^2 + 1\) if and only if \(ab - 1\) divides \(a^2 + b^2 + 3\).
We have
\[
(ab - 1)(a^3b + 3ab + a^2 + 3) = a^4b^2 + 3a^2b^2 + a^3b + 3ab - a^2b - 3ab - a^2 - 3
\]
and so
\[
b^2(a^4 + 3a^2 + 1) - (ab - 1)(a^3b + 3ab + a^2 + 3) = a^2 + b^2 + 3.
\]
Thus if \(ab - 1\) divides \(a^4 + 3a^2 + 1\) then \(ab - 1\) divides \(a^2 + b^2 + 3\). Conversely suppose \(ab - 1\) divides \(a^2 + b^2 + 3\). Then \(ab - 1\) divides \(b^2(a^4 + 3a^2 + 1)\). Since \(ab - 1\) and \(b^2\) are relatively prime we have that \(ab - 1\) divides \(a^4 + 3a^2 + 1\).

Now if \(k > 0\) and \((a, b)\) is a solution to \(a^2 + b^2 + 3 = k(ab - 1)\) then \(b\) is a root of the polynomial \(a^2 + x^2 + 3 = k(ax - 1)\) which can be rewritten as
\[
x^2 - kax + (a^2 + 3 + k) = 0.
\]
Thus if \(b'\) is the other root we have, by Vieta’s formulas, \(b + b' = ka\) and \(bb' = a^2 + 3 + k\). The first shows that \(b'\) is an integer and the second shows that \(b' > 0\). Thus \((a, b')\) is another solution to \(a^2 + b^2 + 3 = k(ab - 1)\).

4) If \(ab - 1\) divides \(a^2 + b^2 + 3\) then \(a^2 + b^2 + 3 = 8(ab - 1)\). Suppose there are positive integers \(a, b, k\) such that \(a^2 + b^2 + 3 = k(ab - 1)\). For this fixed \(k\) let \(S\) be the set of all positive integer pairs \((a, b)\) such that \(a^2 + b^2 + 3 = k(ab - 1)\). Choose an \((a, b)\) \(\in S\) such that \(a + b\) is minimal. Without loss of generality we have \(a \leq b\). Since \(a \neq 1\) we have \(a < b\). Now \((a, b')\) is another solution. Since \(a + b\) is minimal we have \(a + b \leq a + b'\) and hence \(b \leq b'\). Thus
\[
b^2 \leq bb' = a^2 + 3 + k \implies k \geq b^2 - a^2 - 3
\]
and so
\[
a^2 + b^2 + 3 \geq k(ab - 1)
\]
\[
\geq (b^2 - a^2 - 3)(ab - 1)
\]
\[
= ab^3 - b^2 - a^3b + a^2 - 3ab + 3.
\]
Hence
\[
3ab + 2b^2 \geq ab^3 - a^3b \implies 3a + 2b \geq ab^2 - a^3.
\]
Since \(a < b\) we have \(3a + 2b < 5b\) and \(ab^2 - a^3 = a(b + a)(b - a) > ab\). Thus \(5b > ab\) and so \(a < 5\). By 2) the only possible \((a, b)\) are then \((1, 2), (1, 6), \) and \((2, 15)\). Each of these gives \(k = 8\).

Thus 3) and 4) show that our original problem is equivalent to finding all positive integers \(a\) and \(b\) such that \(a^2 + b^2 + 3 = 8(ab - 1)\). We could rewrite this as \((a - 4b)^2 - 15b^2 = -11\) and apply the theory of equations of the form \(x^2 - Dy^2 = N\) as found in, say, section 58 of Nagell’s *Number Theory*. Instead we will determine the solutions by “Vieta jumping” as in the proof of (4).

Let \(S\) be the set of all positive integers pairs \((a, b)\) such that \(a^2 + b^2 + 3 = 8(ab - 1)\). Clearly if \((a, b) \in S\) then \((b, a) \in S\), and, by 1) there are no \((a, b) \in S\) with \(a = b\). Recall that if \((a, b) \in S\) then \((b, b) \in S\) where \(b + b' = 8a\) and \(bb' = a^2 + 11\).

5) For any \((a, b) \in S\) define \(\rho(a, b) = (b', a)\) and \(\lambda(a, b) = (b, 8b - a)\). Then \(\rho(a, b) \in S\), \(\lambda(a, b) \in S\), and \(\lambda(\rho(a, b)) = (a, b)\).
Let \((a, b) \in S\). We have \((a, b') \in S\) and hence \(\rho(a, b) = (b', a) \in S\). Now
\[
    b^2 + (8b - a)^2 + 3 = 64b^2 - 16ab + (a^2 + b^2 + 3)
    = 64b^2 - 16ab + 8(ab - 1)
    = 64b^2 - 8ab - 8
    = 8(b(8b - a) - 1)
\]
so \(\lambda(a, b) = (b, 8b - a) \in S\). Finally,
\[
    \lambda(\rho(a, b)) = \lambda(b', a) = (a, 8a - b')
\]
where
\[
    8a - b' = 8a - \frac{a^2 + 11}{b} = \frac{8ab - a^2 - 11}{b} = \frac{b^2}{b} = b.
\]

6) The only \((a, b) \in S\) such that \(a < b \leq 10\) are \((a, b) = (1, 2)\) and \((1, 6)\).

Since \(a^2 + b^2 + 3 \equiv 0 \mod 8\) we see that \(a\) and \(b\) must have opposite parity and neither can be divisible by 4. Moreover the only such solutions with \(a\) or \(b\) less than 4 are \((1, 2)\) and \((1, 6)\) by 2). This leaves only
\[
    (a, b) = (5, 6), (6, 7), (6, 9), (5, 10), (7, 10), (9, 10)
\]
and none of these satisfy \(a^2 + b^2 + 3 = 8(ab - 1)\).

7) Let \((a, b) \in S\) such that \(b \geq 11\). If \(a < b\) then \(b' < a\).

Suppose first that \(b' \leq 10\). Assume \(a \leq b'\). Since \((a, b') \in S\) we have \(a \neq b'\). Thus \(a < b' \leq 10\). So, by 6), we must have \(a = 1\). But if \(a = 1\) we have \(b = 1\) or \(b = 6\), a contradiction with \(b \geq 11\). Hence \(b' < a\).

Suppose now that \(b' \geq 11\). Again assume \(a \leq b'\). Then, as in the last paragraph, \(a < b'\).

We have
\[
    bb' = a^2 + 11 < (b')^2 + 11 \implies b < b' + \frac{11}{b'} \leq b' + 1
\]
and so \(b \leq b'\). Now swapping \(b\) and \(b'\) we have
\[
    bb' = a^2 + 11 < b^2 + 11 \implies b' < b + \frac{11}{b} \leq b + 1
\]
and so \(b' \leq b\). Thus \(b = b'\). Since \(8a = b + b' = 2b\) we have \(b = 4a\). But then
\[
    a^2 + 16a^2 + 3 = 8(a^2 - 1) \implies 11 = 15a^2,
\]
a contradiction. Hence \(b' < a\).

Finally,

8) \((a, b) \in S\) if and only if \(\{a, b\} = \{s_n, s_{n+1}\}\) or \(\{a, b\} = \{t_n, t_{n+1}\}\) for \(n \geq 0\) where
\[
    s_0 = 1, s_1 = 2, \text{ and } s_n = 8s_{n-1} - s_{n-2} \text{ for } n \geq 2
\]
and
\[
    t_0 = 1, t_1 = 6, \text{ and } t_n = 8t_{n-1} - t_{n-2} \text{ for } n \geq 2.
\]
Note that $\lambda^n(1, 2) = (s_n, s_{n+1})$ and $\lambda^n(1, 6) = (t_n, t_{n+1})$ for all $n \geq 0$.

Since $(1, 2)$ and $(1, 6) \in S$ we see that $(a, b) \in S$ for any $\{a, b\} = \{s_n, s_{n+1}\}$ or $\{a, b\} = \{t_n, t_{n+1}\}$ and $n \geq 0$ by (5).

Now suppose $(a, b) \in S$. Since $(b, a) \in S$ as well, we can suppose without loss of generality that $a < b$. By 5) and 7) there exists an integer $d \geq 0$ such that $\rho^d(a, b) = (a^*, b^*)$ with $a^* < b^* \leq 10$. By (6) we must have $\rho^d(a, b) = (1, 2)$ or $\rho^d(a, b) = (1, 6)$. Since $(a, b) = \lambda^d(\rho^d(a, b))$ we have $(a, b) = \lambda^d(1, 2)$ or $(a, b) = \lambda^d(1, 6)$.

Thus $ab - 1$ divides $a^4 + 3a^2 + 1$ if and only if $a$ and $b$ are consecutive elements of either of the sequences $s_n$ or $t_n$ given above. Since the first few terms of $s_n$ are $1, 2, 15, 118, 929, 7314, 57583, \ldots$ and the first few terms of $t_n$ are $1, 6, 47, 370, 2913, 22934, 180559, \ldots$ the first few solutions to our problem (with $a \leq b$) are

$$ (a, b) = (1, 2), (2, 15), (15, 118), (118, 929), (929, 7314), (7314, 57583), \ldots $$

and

$$ (a, b) = (1, 6), (6, 47), (47, 370), (370, 2913), (2913, 22934), (22934, 180559), \ldots $$

Also solved by Ed Gray, Highland Beach, FL; Kenneth Korbin, New York, NY; Toshihiro Shimizu, Kawasaki, Japan; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

**5436: Proposed by Arkady Alt, San Jose, CA**

Find all values of the parameter $t$ for which the system of inequalities

$$ A = \begin{cases} \sqrt{x} + t \geq 2y \\ \sqrt{y} + t \geq 2z \\ \sqrt{z} + t \geq 2x \end{cases} $$

a) has solutions;
b) has a unique solution.

**Solution by the Proposer**

a) Note that $(A) \iff \begin{cases} t \geq 16y^4 - x \\ t \geq 16z^4 - y \iff 3t \geq 16y^4 - x + 16z^4 - y + 16x^4 - z = \\ t \geq 16x^4 - z \end{cases}$

$$(16x^4 - x) + (16y^4 - y) + (16z^4 - z) \geq 3 \min_{x} (16x^4 - x) \implies t \geq \min_{x} (16x^4 - x).$$

For $x \in \left(0, \frac{1}{16}\right)$, using the AM-GM Inequality, we obtain

$$ x - 16x^4 = x (1 - 16x^3) = \frac{3}{4} x (1 - 16x^3)^{\frac{3}{4}} = \frac{3}{4} \left(\frac{48x^3}{3} \right) \left(\frac{1 - 16x^3}{3}\right)^{\frac{3}{4}} \leq \frac{3}{16} \cdot \left(\frac{3}{4}\right)^{\frac{4}{4}} = \frac{3}{16}. $$

And since $x - 16x^4 \leq 0$ for
then for all \( x \) the inequality \( x - 16x^4 \leq \frac{3}{16} \) holds. Since the upper bound is \( \frac{3}{16} \) for values \( x - 16x^4 \) is attainable when \( x = \frac{1}{4} \), then \( \max (x - 16x^4) = \frac{3}{16} \iff \min_x (16x^4 - x) = -\frac{3}{16} \).

Thus \( t \geq -\frac{3}{16} \) is a necessary condition for the solvability of system \((A)\).

Let’s prove sufficiency.

Let \( t \geq -\frac{3}{16} \). Since function \( h(x) \) is continuous in \( \mathbb{R} \) and \( \min_x (16x^4 - x) = -\frac{3}{16} \), then \( \left[ -\frac{3}{16}, \infty \right) \) is the range of \( h(x) \). This means that for any \( t \geq -\frac{3}{16} \) the equation \( 16x^4 - x = t \) has solution in \( \mathbb{R} \) and since for any \( u \) which is a solution of the equation \( 16x^4 - x = t \), the triple \((x, y, z) = (u, u, u, )\) is a solution of the system \((A)\) then for such \( t \) system \((A)\) solvable as well.

**Remark.**

Actually the latest reasoning about the solvability of system \((A)\) if \( t \geq -\frac{3}{16} \) is redundant for \((a)\) because suffices to note that for such \( t \) the triple \((x, y, z) = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)\) satisfies to \((A)\).

But for \((b)\) criteria of solvability of equation \( 16x^4 - x = t \) in form of inequality \( t \geq -\frac{3}{16} \) is important.

**b)** Note that system \((A)\) always have more the one solution if \( t > -\frac{3}{16} \).

Indeed, let for any \( t_1, t_2 \in \left( -\frac{3}{16}, t \right) \) such that \( t_1 \neq t_2 \) equation \( 16u^4 - u = t_i \) has solution \( u_i, i = 1, 2 \).

Then \( u_1 \neq u_2 \) and two distinct triples \((u_1, u_1, u_1), (u_2, u_2, u_2)\) satisfy to the system \((A)\).

Let \( t = -\frac{3}{16} \). Then \( -\frac{3}{16} \geq 16y^4 - x \iff \frac{3}{16} + x - y \geq 16y^4 - y \geq -\frac{3}{16} \).

Hereof \( x - y \geq 0 \iff x \geq y \). Similarly \( -\frac{3}{16} \geq 16z^4 - y \) and \( -\frac{3}{16} \geq 16x^4 - z \) implies \( y \geq z \) and \( z \geq x \), respectively. Thus in that case \( x = y = z \) and all solutions of the system \((A)\) are represented by solutions of one equation \( 16x^4 - x = -\frac{3}{16} \iff 16x^4 - x + \frac{3}{16} = 0 \iff 256x^4 - 16x + 3 = 0 \) which has only root \( \frac{1}{4} \) because \( 256x^4 - 16x + 3 = (4x - 1)^2 (16x^2 + 8x + 3) \).

Thus, system \((A)\) has unique solution iff \( t = \frac{1}{4} \).

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David Stone and John Hawkins, Georgia.
Let \( f : C - \{2\} \to C \) be the function defined by \( f(z) = \frac{2 - 3z}{z - 2} \). If \( f^n(z) = (f \circ f \circ \ldots \circ f)(z) \), then compute \( f^n(z) \) and \( \lim_{n \to +\infty} f^n(z) \).

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Assume first that \( z \neq 2 \) and \( f^n(z) \) exists for all \( n \geq 1 \). Then, direct computation yields

\[
    f^2(z) = \frac{10 - 11z}{5z - 6} \quad \text{and} \quad f^3(z) = \frac{42 - 43z}{21z - 22}.
\]

(1)

When these are combined with the formula for \( f(z) \), it appears that there is a sequence \( \{x_n\} \) of positive integers such that

\[
    f^n(z) = \frac{2x_n - (2x_n + 1)z}{x_nz - (x_n + 1)}
\]

(2)

for all \( n \geq 1 \). Since \( f(z) = \frac{2 - 3z}{z - 2} \), we have \( x_1 = 1 \). Further, if (2) holds for some \( n \geq 1 \), then

\[
    f^{n+1}(z) = f(f^n(z))
\]

\[
    = \frac{2 - 3f^n(z)}{f^n(z) - 2}
\]

\[
    = \frac{2 - 3 \left[ \frac{2x_n - (2x_n + 1)z}{x_nz - (x_n + 1)} \right]}{\left[ \frac{2x_n - (2x_n + 1)z}{x_nz - (x_n + 1)} \right] - 2}
\]

\[
    = \frac{2 \left[ x_nz - (x_n + 1) \right] - 3 \left[ 2x_n - (2x_n + 1)z \right]}{\left[ 2x_n - (2x_n + 1)z \right] - 2 \left[ x_nz - (x_n + 1) \right]}
\]

\[
    = \frac{(8x_n + 2) - (8x_n + 3)z}{(4x_n + 1)z - (4x_n + 2)}. \tag*{(4)}
\]

This suggests that \( x_{n+1} = 4x_n + 1 \) for \( n \geq 1 \). These conditions on \( \{x_n\} \) are consistent with the formula for \( f(z) \) and property (2). Note finally that

\[
    x_1 = 1 = \frac{3}{3} = \frac{4 - 1}{3}, \quad x_2 = 5 = \frac{15}{3} = \frac{4^2 - 1}{3}, \quad \text{and} \quad x_3 = 21 = \frac{63}{3} = \frac{4^3 - 1}{3}.
\]
This leads us to conjecture that \( x_n = \frac{4^n - 1}{3} \) and hence,

\[
f^n(z) = 2 \left( \frac{4^n - 1}{3} \right) - \left[ 2 \left( \frac{4^n - 1}{3} \right) + 1 \right] \frac{z}{(4^n - 1) z - (4^n + 2)} = 2 \left( \frac{4^n - 1}{3} \right) - \left( 2 \cdot 4^n + 1 \right) \frac{z}{(4^n - 1) z - (4^n + 2)}
\]

for all \( n \geq 1 \).

If \( f^n(z) \) exists for all \( n \geq 1 \), let \( P(n) \) be the statement

\[
f^n(z) = 2 \left( \frac{4^n - 1}{3} \right) - \left( 2 \cdot 4^n + 1 \right) \frac{z}{(4^n - 1) z - (4^n + 2)}.
\]

If \( n = 1 \),

\[
\frac{2 (4 - 1) - (2 \cdot 4 + 1) z}{(4 - 1) z - (4 + 2)} = \frac{6 - 9z}{3z - 6} = \frac{2 - 3z}{z - 2}
\]

and thus, \( P(1) \) is true. Assume that \( P(n) \) is true, i.e.,

\[
f^n(z) = 2 \left( \frac{4^n - 1}{3} \right) - \left( 2 \cdot 4^n + 1 \right) \frac{z}{(4^n - 1) z - (4^n + 2)}
\]

for some \( n \geq 1 \). Then,

\[
f^{n+1}(z) = f(f^n(z))
\]

\[
= 2 - 3 \left[ 2 \left( \frac{4^n - 1}{3} \right) - \left( 2 \cdot 4^n + 1 \right) \frac{z}{(4^n - 1) z - (4^n + 2)} \right] - 2
\]

\[
= 2 \left[ (4^n - 1) z - (4^n + 2) \right] - 3 \left[ 2 \left( \frac{4^n - 1}{3} \right) - \left( 2 \cdot 4^n + 1 \right) \frac{z}{(4^n - 1) z - (4^n + 2)} \right] - 2 \left[ (4^n - 1) z - (4^n + 2) \right]
\]

\[
= \frac{2 (4^n - 1) + 3 \left( 2 \cdot 4^n + 1 \right) z - [2 (4^n + 2) + 6 (4^n - 1) \left( 2 \cdot 4^n + 1 \right) z]}{2 (4^n - 1) + 2 (4^n + 2) - [2 \cdot 4^n + 1 + 2 (4^n - 1) \left( 2 \cdot 4^n + 1 \right) z]}
\]

\[
= \frac{\left( 2 \cdot 4^{n+1} + 1 \right) z - 2 \left( 4^{n+1} + 1 \right) \left( 4^{n+1} + 2 \right)}{(4^{n+1} - 1) z - (4^{n+1} + 2)}
\]

and therefore, \( P(n + 1) \) is also true. By Mathematical Induction, \( P(n) \) is true for all \( n \geq 1 \).

Because formula (3) required the assumption that \( f^n(z) \) exists for all \( n \geq 1 \), we need to determine if there are points \( z \in C \setminus \{2\} \) for which there is a positive integer \( m \) such that
If \( f^n(z) \) does not exist for \( n > m \). The existence of \( f^n(z) \) requires that \( z, f(z), \ldots, f^{n-1}(z) \neq 2 \). Therefore, we have to find all points \( z \) for which \( f^m(z) = 2 \) for some \( m \geq 1 \). One way to do this is to consider the inverse function

\[
f^{-1}(z) = \frac{2z + 2}{z + 3}
\]

and describe

\[
f^{-m}(z) = \left( f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1} \right)(z)
\]

in a manner similar to that used to find formula (3). If we do so, we see that for \( z \neq -3 \),

\[
f^{-m}(z) = \frac{(4^m + 2)z + 2 (4^m - 1)}{(4^m - 1)z + 2 \cdot 4^m + 1}.
\]

In particular,

\[
f^{-m}(2) = \frac{(4^m + 2) \cdot 2 + 2 (4^m - 1)}{(4^m - 1) \cdot 2 + 2 \cdot 4^m + 1} = \frac{4^{m+1} + 2}{4^{m+1} - 1}.
\]

If \( z_m = \frac{4^{m+1} + 2}{4^{m+1} - 1} \) for some \( m \geq 1 \), then it follows that \( f^m(z_m) = 2 \) and hence, \( f^n(z_m) \) is undefined for \( n > m \). Therefore, \( \lim_{n \to +\infty} f^n(z_m) \) does not exist for these points.

Let

\[
S = \{ 2 \} \cup \left\{ \frac{4^{m+1} + 2}{4^{m+1} - 1} : m \in N \right\}.
\]

For \( z \notin S \), \( f^n(z) \) exists for all \( n \geq 1 \). If \( z = 1 \), then \( z \notin S \) and (3) implies that

\[
f^n(1) = \frac{2 (4^n - 1) - (2 \cdot 4^n + 1)}{(4^n - 1) - (4^n + 2)}
\]

\[
= -\frac{3}{-3} = 1
\]

for all \( n \geq 1 \). Hence, \( \lim_{n \to +\infty} f^n(1) = 1 \). For all other values of \( z \notin S \),

\[
\lim_{n \to +\infty} f^n(z) = \lim_{n \to +\infty} \frac{2 (4^n - 1) - (2 \cdot 4^n + 1) z}{(4^n - 1) z - (4^n + 2)}
\]

\[
= \lim_{n \to +\infty} \frac{2 (1 - 4^{-n}) - (2 + 4^{-n}) z}{(1 - 4^{-n}) z - (1 + 2 \cdot 4^{-n})}
\]

\[
= \frac{2 - 2z}{z - 1} = -2.
\]

Therefore, for \( z \notin S \),

\[
\lim_{n \to +\infty} f^n(z) = \begin{cases} 
1 & \text{if } z = 1 \\
-2 & \text{otherwise}
\end{cases}
\]

Solution 2 by Henry Ricardo, Westchester Math Circle, NY
We take advantage of the well-known homomorphism between $2 \times 2$ matrices and Möbius transformations: \[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow f(z) = \frac{az + b}{cz + d} \] In this relation, the $n$-fold composition $f^n(z)$ corresponds to the $n$th power of $A$. Here we are dealing with powers of the matrix $A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$.

Now we invoke a known result that is a consequence of the Cayley-Hamilton theorem: If $A \in M_2(\mathbb{C})$ and the eigenvalues $\lambda_1, \lambda_2$ of $A$ are not equal, then for all $n \geq 1$ we have

$$A^n = \lambda_1^n B + \lambda_2^n C,$$

where $B = \frac{1}{\lambda_1 - \lambda_2} (A - \lambda_2 I_2)$ and $C = \frac{1}{\lambda_2 - \lambda_1} (A - \lambda_1 I_2)$.

(See, for example, Theorem 2.25(a) in *Essential Linear Algebra with Applications* by T. Andreescu, Birkhäuser, 2014.)

The eigenvalues of the given matrix $A$ are $-1$ and $-4$, so we apply ($\ast$) to get

$$A^n = \frac{(-1)^n}{3} (A + 4I_2) - \frac{(-4)^n}{3} (A + I_2) = \left( \frac{(-1)^n - (-4)^n}{3} \right) A + \left( \frac{4 \cdot (-1)^n - (-4)^n}{3} \right) I_2 = \begin{pmatrix} \frac{1}{3}(-1)^n(1 + 2 \cdot 4^n) & \frac{2}{3}(-1)^n + \frac{2}{3}(-1)^{n+1}4^n \\ \frac{1}{3}(-1)^n + \frac{1}{3}(-1)^{n+1}4^n & \frac{1}{3}(-1)^n(2 + 4^n) \end{pmatrix}.$$ 

After some simplification, we see that

$$f^n(z) = \frac{(2 \cdot 4^n + 1)z - 2(4^n - 1)}{(1 - 4^n)z + (4^n + 2)}.$$ 

Finally, we note that $f^n(1) = 3/3 = 1$; and, for $z \neq 1$, we have

$$\lim_{n \to +\infty} f^n(z) = \lim_{n \to +\infty} \frac{(2 \cdot 4^n + 1)z - 2(4^n - 1)}{(1 - 4^n)z + (4^n + 2)} = \frac{2(z - 1)}{1 - z} = -2.$$ 

Therefore,

$$\lim_{n \to +\infty} f^n(z) = \begin{cases} 1 & \text{if } z = 1, \\ -2 & \text{if } z \neq 1. \end{cases}$$

**Solution 3 by David E. Manes, Oneonta, NY**

We will show by induction that

$$f^{(n)}(z) = \frac{2 - \frac{2a_n + 1}{z}}{\frac{a_n}{z} - \frac{a_n + 1}{a_n}}$$

where $a_n = 4^n - 1$. If $n = 1$, then $a_1 = 1$ and $f^{(1)}(z) = \frac{(2 - 3z)}{(z - 2)} = f(z)$. Therefore, the result is true for $n = 1$. Assume the positive integer $n \geq 1$ and the given formula is valid.
for \( f^{(n)}(z) \). Then

\[
f^{(n+1)}(z) = f(f^{(n)}(z)) = 2 - 3 \left( \frac{2 - 2a_n + 1}{a_n} \right) \left( \frac{z - a_n + 1}{z - a_n} \right)
\]

\[
= 2a_n z - 2a_n - z - 2a_n z + 2a_n + 2 = -2 - 8a_n + (8a_n + 3)z
\]

\[
= \frac{2 + 8a_n - (8a_n + 3)z}{(4a_n + 1)z - (4n + 2)} = \frac{2 + 8 \left( \frac{4^n - 1}{3} \right) - \left( 8 \left( \frac{4^n - 1}{3} \right) + 3 \right) z}{4 \left( \frac{4^n - 1}{3} \right) + 1} - \left( \frac{4^n - 1}{3} \right) + 2
\]

\[
= \frac{-2 + 2 \cdot 4^{n+1} - (1 + 2 \cdot 4^{n+1})z}{(4^{n+1} - 1)z - (4^{n+1} + 2)}
\]

\[
= 2 - \left( \frac{2 \cdot 4^{n+1} + 1}{4^{n+1} - 1} \right) z - \left( \frac{4^{n+1} + 2}{4^{n+1} - 1} \right)
\]

\[
= \frac{2 - \left( 2a_{n+1} + 1 \right)}{a_{n+1} + 1}
\]

where \( a_{n+1} = \frac{4^{n+1} - 1}{3} \). Note that \( \frac{4^{n+1} + 2}{3} = \frac{4^{n+1} - 1}{3} + 1 = a_{n+1} + 1 \) and

\[
\frac{2 \cdot 4^{n+1} + 1}{3} = \frac{2 \cdot 4^{n+1} - 2}{3} + 1 = 2 \left( \frac{4^{n+1} - 1}{3} \right) + 1 = 2a_{n+1} + 1.
\]

Hence, the result is true for the integer \( n + 1 \) so that by the principle of mathematical induction the result is valid for all positive integers \( n \).

For the limit question, note that if \( f(z) = z \), then \( z = 1 \) or \( z = -2 \). Therefore, one of the fixed points of \( f \) is \( z = 1 \) so that \( f^{(n)}(1) = 1 \) for each positive integer \( n \) and \( \lim_{n \to +\infty} f^{(n)}(1) = 1 \). Moreover, observe that

\[
\lim_{n \to +\infty} \frac{1}{a_n} = \lim_{n \to +\infty} \frac{3}{4^n - 1} = 0.
\]

Therefore, if \( z \neq 1 \), then
\[
\lim_{n \to +\infty} f^{(n)}(z) = \lim_{n \to +\infty} \left( 2 - \frac{2a_n + 1}{a_n} \right) z - \frac{a_n + 1}{a_n} = \left( 2 - \lim_{n \to +\infty} \left( 2 + \frac{1}{a_n} \right) \right) z = \frac{2 - 2z}{z - 1} = -2.
\]

Hence,
\[
\lim_{n \to +\infty} f^{(n)}(z) = \begin{cases} 
1, & \text{if } z = 1, \\
-2, & \text{if } z \neq 1.
\end{cases}
\]

Solution 4 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Recall the map 
\[
f(z) = \frac{az + b}{cz + d} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
gives a group isomorphism between group of fractional linear transformations
\[
\left\{ f : f(z) = \frac{az + b}{cz + d} \text{ where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}
\]
under function composition and the group
\[
GL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}
\]
derived under matrix multiplication.

To compute \( f^n(z) \), let \( M = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \). Using induction, we show

\[
M^n = \frac{(-1)^n}{3} \begin{pmatrix} 2^{2n+1} + 1 & -2^{2n+1} + 2 \\ -4^n + 1 & 4^n + 2 \end{pmatrix}
\]

Observe \( M^1 = \frac{-1}{3} \begin{pmatrix} 2^3 + 1 & -2^3 + 2 \\ -3 & 6 \end{pmatrix} = \frac{-1}{3} \begin{pmatrix} 9 & -6 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -1 & -2 \end{pmatrix} \).

Assume

\[
M^n = \frac{(-1)^n}{3} \begin{pmatrix} 2^{2n+1} + 1 & -2^{2n+1} + 2 \\ -4^n + 1 & 4^n + 2 \end{pmatrix}
\]

and observe

\[
M^{n+1} = M^n M
= \frac{(-1)^n}{3} \begin{pmatrix} 2^{2n+1} + 1 & -2^{2n+1} + 2 \\ -4^n + 1 & 4^n + 2 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}
= \frac{(-1)^n}{3} \begin{pmatrix} -3(2^{2n+1} + 1) + (-2^{2n+1} + 2) & 2(2^{2n+1} + 1) - 2(-2^{2n+1} + 2) \\ -3(-4^n + 1) + (4^n + 2) & 2(-4^n + 1) - 2(4^n + 2) \end{pmatrix}
= \frac{(-1)^{n+1}}{3} \begin{pmatrix} 2^{2(n+1)+1} + 1 & -2^{2(n+1)+1} + 2 \\ -4^{n+1} + 1 & 4^{n+1} + 2 \end{pmatrix}
\]

Using the aforementioned group isomorphism and simplifying, we conclude

\[
f^n(z) = \frac{(2^{2n+1} + 1)z - 2^{2n+1} + 2}{(-4^n + 1)z + 4^n + 2} = \frac{(2 \cdot 4^n + 1)z + (2 - 2 \cdot 4^n)}{(1 - 4^n)z + (2 + 4^n)}.
\]
Notice that the map $f^n(z)$ is undefined for $z = \frac{4^k + 2}{4^k - 1}$ where $1 \leq k \leq n$. Consequently, $\lim_{n \to +\infty} f(z)$ does not exist for these values of $z$. Furthermore,

$$\lim_{n \to +\infty} f^n(z) = \lim_{n \to +\infty} \frac{(2 \cdot 4^n + 1)z + (2 - 2 \cdot 4^n)}{(1 - 4^n)z + (2 + 4^n)} = \lim_{n \to +\infty} \frac{(2 + \frac{1}{4^n})z + \left(\frac{2}{4^n} - 2\right)}{\left(\frac{1}{4^n} - 1\right)z + \left(\frac{2}{4^n} + 1\right)} \quad = \frac{2z - 2}{-z + 1} \quad = -2 \left(\frac{1 - z}{1 - z}\right).$$

Note $f(1) = 1$ so $f^n(1) = 1$ for all $n \geq 1$. It follows that

$$\lim_{n \to +\infty} f(z) = \begin{cases} \text{DNE} & \text{if } z = \frac{4^n + 2}{4^n - 1} \text{ where } n \in \mathbb{Z}_{>0} \\ 1 & \text{if } z = 1 \\ -2 & \text{otherwise.} \end{cases}$$

**Comment by Editor:** David Stone and John Hawkins of Georgia Southern University stated the following in their solution: “The appearance of so many sums of powers of 4 prompts us to offer a candidate for the cutest representation of $f^n(z)$:

$$f^n(z) = \frac{(2 \cdot 111 \ldots, 14 + 1)z - 2 \cdot 111 \ldots, 14}{-111 \ldots, 14z + (111 \ldots, 14 + 1)},$$

where each of the base 4 repunits has $n - 1$ digits.”

**Solution 5 by Toshihiro Shimizu, Kawasaki, Japan**

Let $f^n(z) = \frac{a_nz + b_n}{c_nz + d_n}$. Then, we have

$$\frac{a_{n+1}z + b_{n+1}}{c_{n+1}z + d_{n+1}} = f^{n+1}(z) = f^n \left(\frac{2 - 3z}{z - 2}\right) = \frac{(b_n - 3a_n)z + 2(a_n - b_n)}{(d_n - 3c_n)z + 2(c_n - d_n)}$$

Therefore, we have $a_{n+1} = b_n - 3a_n$, $b_{n+1} = 2a_n - 2b_n$ and $c_{n+1} = d_n - 3c_n$, $d_{n+1} = 2c_n - 2d_n$. Since $f^0(z) = z$, $a_0 = 1$, $b_0 = c_0 = 0$ and $d_0 = 1$. Since $b_n = a_{n+1} + 3a_n$, we have

$$a_{n+2} + 3a_{n+1} = 2a_n - 2(a_{n+1} + 3a_n)$$

$$a_{n+2} + 5a_{n+1} + 4a_n = 0$$
and \( a_1 = b_0 - 3a_0 = -3 \). Thus, we have

\[
a_n = \frac{1}{3} (-1)^n + \frac{2}{3} (-4)^n
\]

\[
b_n = a_{n+1} + 3a_n = \frac{1}{3} (-1)^{n+1} + \frac{2}{3} (-4)^{n+1} + (-1)^n + 2 (-4)^n
\]

\[
= \frac{2}{3} (-1)^n - \frac{2}{3} (-4)^n.
\]

Similarly, we have \( c_{n+2} + 5c_{n+1} + 4c_n = 0 \) and \( c_1 = d_0 - 3c_0 = 1 \). Thus, we have

\[
c_n = \frac{1}{3} (-1)^n - \frac{1}{3} (-4)^n
\]

\[
d_n = c_{n+1} + 3c_n
\]

\[
= \frac{2}{3} (-1)^n + \frac{1}{3} (-4)^n
\]

Therefore,

\[
f^n(z) = \frac{((-1)^n + 2 (-4)^n) z + (2 (-1)^n - 2 (-4)^n)}{((-1)^n - (-4)^n) z + (2 (-1)^n + (-4)^n)}.
\]

If \( z \neq 1 \), we have

\[
f^n(z) = \frac{((\frac{1}{4})^n + 2) z + (2 (\frac{1}{4})^n - 2)}{((\frac{1}{4})^n - 1) z + (2 (\frac{1}{4})^n + 1)}
\]

\[
\rightarrow \frac{2z - 2}{-z + 1}
\]

\[
= -2 \quad (n \rightarrow +\infty).
\]

If \( z = 1 \), the value of \( f^n(z) \) is always 1 and its limit is also 1.

**Solution 6 by Kee-Wai Lau, Hong Kong, China**

It can easily be proved by induction that

\[
f^n(z) = \frac{2(2^{2n} - 1) - (2^{2n+1} + 1)z}{(2^{2n} - 1)z - (2^{2n} + 1)},
\]

whenever \( z \not\in S_n \), where \( S_n = \{2\} \cup \left\{ \frac{2(2^{2k-1} + 1)}{2^{2k} - 1} : k = 1, 2, 3, \ldots, n \right\} \).

Clearly, \( \lim_{n \rightarrow \infty} f^n(1) = 1 \) and if \( z \not\in T \), where \( T = \{1, 2\} \cup \left\{ \frac{2(2^{2k-1} + 1)}{2^{2k} - 1}, k = 1, 2, 3, \ldots \right\} \), then \( \lim_{n \rightarrow \infty} f^n(z) = -2 \).

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Brian D. Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy (two solutions), Rehovot, Israel; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.
5438: Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

Let $k \geq 0$ be an integer and let $\alpha > 0$ be a real number. Prove that

$$\frac{x^{2k}}{(1 - x^2)^{\alpha}} + \frac{y^{2k}}{(1 - y^2)^{\alpha}} + \frac{z^{2k}}{(1 - z^2)^{\alpha}} \geq \frac{x^k y^k}{(1 - xy)^{\alpha}} + \frac{y^k z^k}{(1 - yz)^{\alpha}} + \frac{x^k z^k}{(1 - xz)^{\alpha}},$$

for $x, y, z \in (-1, 1)$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that by the Binomial theorem,

$$\frac{t^{2k}}{(1 - t^2)^{\alpha}} = \sum_{j=0}^{\infty} \left( \begin{array}{c} -\alpha \\ j \end{array} \right) t^{2j}, -1 < t < 1,$$

where $\binom{-\alpha}{j} = \frac{-\alpha(\alpha + 1) \cdots (\alpha + j - 1)}{j!} > 0$ for all indices $j \geq 0$.

Therefore, by the AM–GM inequality,

$$\frac{x^{2k}}{(1 - x^2)^{\alpha}} + \frac{y^{2k}}{(1 - y^2)^{\alpha}} + \frac{z^{2k}}{(1 - z^2)^{\alpha}} = \frac{1}{2} \sum_{cyc} \left( \frac{x^{2k}}{(1 - x^2)^{\alpha}} + \frac{y^{2k}}{(1 - y^2)^{\alpha}} \right)$$

$$\geq \sum_{cyc} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} (xy)^{k+j},$$

as claimed.

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well known that for any real numbers $a, b, c$

$$a^2 + b^2 + c^2 \geq ab + bc + ca. \quad (1)$$

We show that $a, b \in (-1, 1)$

$$\sqrt{(1 - a^2)(1 - b^2)} \leq 1 - ab. \quad (2)$$

Suppose that to the contrary $\sqrt{(1 - a^2)(1 - b^2)} > 1 - ab$, by squaring both sides of the inequality, we get $1 - a^2 - b^2 + a^2 b^2 > 1 - 2ab + a^2 b^2$, which implies that
\(-a^2 - b^2 + 2ab = -(a - b)^2 > 0\), which is impossible, that is, (2) is proved. From (2), we can conclude that
\[
\frac{1}{\sqrt{(1-a^2)(1-b^2)}} \geq \frac{1}{1-ab}.
\tag{3}
\]
Now, using (1) and (3), we write
\[
x^{2k + (1-x^2)\alpha} + y^{2k + (1-y^2)\alpha} + z^{2k + (1-z^2)\alpha} \geq \frac{x^k y^k}{(1-x^2)(1-y^2)} + \frac{y^k z^k}{(1-y^2)(1-z^2)} + \frac{z^k x^k}{(1-z^2)(1-x^2)}
\]
\[= \frac{x^k y^k}{\sqrt{(1-x^2)(1-y^2)}} + \frac{y^k z^k}{\sqrt{(1-y^2)(1-z^2)}} + \frac{z^k x^k}{\sqrt{(1-z^2)(1-x^2)}} \geq \frac{x^k y^k}{(1-xy)^\alpha} + \frac{y^k z^k}{(1-yz)^\alpha} + \frac{z^k x^k}{(1-zx)^\alpha}.
\]

**Solution 3 by Moti Levy, Rehovot, Israel**

Since
\[
|a|^k \geq \frac{a^k}{(1-a)^\alpha}, \quad a \in (-1, 1)
\]
then
\[
\frac{|x|^k |y|^k}{(1-|x||y|)^\alpha} + \frac{|y|^k |z|^k}{(1-|y||z|)^\alpha} + \frac{|z|^k |x|^k}{(1-|z||x|)^\alpha} \geq \frac{x^k y^k}{(1-xy)^\alpha} + \frac{y^k z^k}{(1-yz)^\alpha} + \frac{z^k x^k}{(1-zx)^\alpha}.
\]

Therefore, we can assume that \(x, y, z \in (0, 1)\). Using the generalized binomial theorem,
\[
\frac{1}{(1-u)^\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} u^n, \quad |u| < 1.
\]

By the inequality \(a^2 + b^2 + c^2 \geq ab + bc + ca, \quad a, b, c \geq 0\),
\[
(x^{n+k})^2 + (y^{n+k})^2 + (z^{n+k})^2 \geq x^{n+k} y^{n+k} + y^{n+k} z^{n+k} + z^{n+k} x^{n+k}.
\]
\[
\frac{x^{2k}}{(1 - x^2)\alpha} + \frac{y^{2k}}{(1 - y^2)\alpha} + \frac{z^{2k}}{(1 - z^2)\alpha}
= \sum_{n=0}^{\infty} \frac{\Gamma(n + a)}{n!\Gamma(\alpha)} x^{2(n+k)} + \sum_{n=0}^{\infty} \frac{\Gamma(n + a)}{n!\Gamma(\alpha)} y^{2(n+k)} + \sum_{n=0}^{\infty} \frac{\Gamma(n + a)}{n!\Gamma(\alpha)} z^{2(n+k)}
= \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\Gamma(n+k)}{n!\Gamma(\alpha)} (x^{n+k} + y^{n+k} + z^{n+k})
\geq \sum_{n=0}^{\infty} \frac{\Gamma(n + a)}{n!\Gamma(\alpha)} \left(x^{n+k} + y^{n+k} + z^{n+k}\right)
\]

\[
= \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\Gamma(n+k)}{n!\Gamma(\alpha)} x^{n+k} + \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\Gamma(n+k)}{n!\Gamma(\alpha)} y^{n+k} + \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\Gamma(n+k)}{n!\Gamma(\alpha)} z^{n+k}
= \frac{x^k y^k}{(1 - xy)^\alpha} + \frac{y^k z^k}{(1 - yz)^\alpha} + \frac{z^k x^k}{(1 - zx)^\alpha}.
\]

**Solution 4 by Kee-Wai Lau, Hong Kong, China**

We first note that

\[
0 < (1 - x^2)(1 - y^2) = (1 - xy)^2 - (x - y)^2 \leq (1 - x)^2.
\]

Hence by the AM-GM inequality, we have

\[
\frac{x^{2k}}{(1 - x^2)\alpha} + \frac{y^{2k}}{(1 - y^2)\alpha} \geq \frac{2|xy|^k}{\sqrt{(1 - x^2)\alpha(1 - y^2)\alpha}} \geq \frac{2|xy|^k}{(1 - xy)^\alpha}.
\]

Similarly,

\[
\frac{y^{2k}}{(1 - y^2)\alpha} + \frac{z^{2k}}{(1 - z^2)\alpha} \geq \frac{2|yz|^k}{(1 - yz)^\alpha} \quad \text{and}
\]

\[
\frac{z^{2k}}{(1 - z^2)\alpha} + \frac{x^{2k}}{(1 - x^2)\alpha} \geq \frac{2|xz|^k}{(1 - zx)^\alpha}.
\]

Adding these inequalities, we easily deduce the inequality of the problem.

*Also solved by Ed Gray, Highland Beach, FL; Nicusor Zlota, “Traian Vuia” Technical College, Focsani, Romania; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.*

**Mea Culpa**

For a variety of reasons, mostly caused by sloppy bookkeeping, those listed below were not credited for having solved the following problems, but should have been.
5427: Paul M. Harms, North Newton, KS.

5428: Ed Gray, Highland Beach, FL; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

5429: Brian D. Beasley, Presbyterian College, Clinton, SC.

5431: Albert Stadler, Herrliberg, Switzerland.