This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2018

• **5469**: Proposed by Kenneth Korbin, New York, NY

Let \( x \) and \( y \) be positive integers that satisfy the equation \( 3x^2 = 7y^2 + 17 \). Find a pair of larger integers that satisfy this equation expressed in terms of \( x \) and \( y \).


Prove that there are an infinite number of Heronian triangles (triangles whose sides and area are natural numbers), whose side lengths are three consecutive natural numbers.

• **5471**: Proposed by Arkady Alt, San Jose, CA

For natural numbers \( p \) and \( n \) where \( n \geq 3 \) prove that
\[
\frac{n}{p} > (n + p)\frac{1}{(n+1)(n+2)(n+3)\cdots(n+p)}.
\]

• **5472**: Proposed by Francisco Perdomo and Ángel Plaza, both at Universidad Las Palmas de Gran Canaria, Spain

Let \( \alpha, \beta, \) and \( \gamma \) be the three angles in a non-right triangle. Prove that
\[
\frac{1 + \sin^2 \alpha}{\cos^2 \alpha} + \frac{1 + \sin^2 \beta}{\cos^2 \beta} + \frac{1 + \sin^2 \gamma}{\cos^2 \gamma} \geq \frac{1 + \sin \alpha \sin \beta}{1 - \sin \alpha \sin \beta} + \frac{1 + \sin \beta \sin \gamma}{1 - \sin \beta \sin \gamma} + \frac{1 + \sin \gamma \sin \alpha}{1 - \sin \gamma \sin \alpha}.
\]

• **5473**: Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Let \( x_1, \ldots, x_n \) be positive real numbers. Prove that for \( n \geq 2 \), the following inequality holds:
\[
\left( \sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}} \right) \left( \sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}} \right) \leq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{x_k}.
\]
(Here the subscripts are taken modulo \( n \))
• **5474**: Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

Let \( a, b \in \mathbb{R}, b \neq 0 \). Calculate

\[
\lim_{n \to \infty} \left( 1 - \frac{a}{n^2} \cdot \frac{b}{n} + \frac{a}{n^2} \right)^n.
\]

**Solutions**

• **5451**: Proposed by Kenneth Korbin, New York, NY

Given triangle \( ABC \) with sides \( a = 8, b = 19 \) and \( c = 22 \). The triangle has an interior point \( P \) where \( AP, BP, \) and \( CP \) each have positive integer length. Find \( AP \) and \( BP \), if \( CP = 4 \).

**Solution 1 by David E. Manes, Oneonta, NY**

We will show that \( BP = 6 \) and \( AP = 17 \).

Using the law of cosines in \( \Delta ABC \), one obtains

\[
\cos \angle C = \frac{8^2 + 4^2 - 22^2}{2 \cdot 8 \cdot 19} = \frac{-59}{304}
\]

so that \( \angle C = \arccos \left( \frac{-59}{304} \right) \). Let \( x = BP \) and \( y = AP \). By the triangle inequality in \( \Delta PCB \), it follows that \( 5 \leq x \leq 11 \). If \( x = 5 \), then

\[
\cos \angle BCP = \frac{8^2 + 4^2 - 5^2}{2 \cdot 8 \cdot 4} = \frac{55}{64}.
\]

Therefore, \( \angle BCP = \arccos \left( \frac{55}{64} \right) \) and

\[
\angle PCA = \angle C - \angle BCP = \arccos \left( \frac{-59}{304} \right) - \arccos \left( \frac{55}{64} \right).
\]

Using the identity \( \cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta \), we get

\[
\cos \angle PCA = \left( \frac{-59}{304} \right) \left( \frac{55}{64} \right) + \left( \frac{77\sqrt{15}}{304} \right) \left( \frac{3\sqrt{7} \cdot 17}{64} \right) = \frac{-3245 + 231\sqrt{3} \cdot 5 \cdot 7 \cdot 17}{304 \cdot 64}.
\]

Thus,

\[
y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19 \cos \angle PCA = 377 - 19 \left( \frac{-3245 + 231\sqrt{15} \cdot 119}{304 \cdot 8} \right)
\]

\[
y = \sqrt{916864 + 61655 - 4389\sqrt{1785}}
\]

Therefore,

\[
y \approx 18.058
\]
is not an integer. Hence, \( x \neq 5 \).

However, if \( x = 6 \), then

\[
\cos \angle BCP = \frac{8^2 + 4^2 - 6^2}{64} = \frac{11}{16}
\]

so that \( \angle BCP = \arccos \left( \frac{11}{16} \right) \) and

\[
\angle PCA = \angle C - \angle BCP = \arccos \left( \frac{-59}{304} \right) - \arccos \left( \frac{11}{16} \right).
\]

Thus,

\[
\cos \angle PCA = \cos \left[ \arccos \left( \frac{-59}{304} \right) - \arccos \left( \frac{11}{16} \right) \right]
\]

\[
= \left( \frac{-59}{304} \right) \left( \frac{11}{16} \right) + \left( \frac{77\sqrt{15}}{304} \right) \left( \frac{3\sqrt{15}}{16} \right) = \frac{-649 + 3465}{4864}
\]

\[
= \frac{11}{19}.
\]

Therefore,

\[
y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19 \left( \frac{11}{19} \right) = 289
\]

whence \( y = 17 \). Hence, \( x = \overline{BP} = 6 \) and \( y = \overline{AP} = 17 \). The solution is unique since \( x = 7 \) does not yield an integer value for \( y \) while each of the values \( x = 8, 9, 10, 11 \) does not yield a triangle for \( \triangle BPA \).

**Solution 2 by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel**

Let \( \overline{BP} = x \) and \( \overline{AP} = y \). Because of the triangle inequality, \( 8 < x + 4, x < 8 + 4 \) or \( 5 \leq x \leq 11 \). Similarly, we have \( 16 \leq y \leq 22 \).
These inequalities can be improved slightly using Stewart’s formula for the length of cevians: if $ABC$ is a triangle with sides $\overline{AC} = b$ and $\overline{BC} = a$ and if $d$ is the length of a cevian from $A$ which divides $\overline{AB}$ into segments of lengths $\overline{AB} = m$ and $\overline{AB} = n$, then:

$$d^2 = \frac{ma^2 + nb^2}{m + n} - mn$$

(this is just an easy consequence of the law of cosines). Since the maximum value of $y$ occurs when $P$ lies on $\overline{BC}$, by Stewart’s formula, $y_{\text{max}}^2 = \frac{22^2 + 19^2}{2} - 4^2 = 406.5 = 20.16^2$, so $y \leq 20$. Similarly, the maximum value of $x$ occurs when $P$ lies on $\overline{AC}$, so that $P$ divides $\overline{AC}$ into segments of lengths 4 and 15. Thus, again by Stewart’s formula $x_{\text{max}}^2 = \frac{15 \cdot 8^2 + 4 \cdot 22^2}{19} - 4 \cdot 15 = 92.42 \approx 9.61^2$, so that $x \leq 9$. Hence:

$$5 \leq x \leq 9$$
$$16 \leq y \leq 20$$

Since $P$ lies on a circle centered at $C$, and the lines $\overline{BP}$ all lie on one side of $\overline{BC}$, each length $x$ of $\overline{BP}$ corresponds to a unique $P$ and, therefore, to a unique value of $y$.

To find $y$ for a given value of $x$, let $\angle BCP = \theta$, $\angle PCA = \phi$, and $\angle BCA = \gamma$. The cosine of $\gamma$ is fixed and given by the law of cosines:

$$\cos \gamma = \frac{19^2 + 8^2 - 22^2}{2 \cdot 8 \cdot 19} = -\frac{59}{304}$$

The sine of $\gamma$ is just $\sqrt{1 - \cos^2 \gamma}$, that is:

$$\sin \gamma = \sqrt{1 - \frac{59^2}{304^2}} = \frac{77}{304} \sqrt{15}$$

The cosine of $\theta$ for a given value of $x$ is also given by the law of cosines:

$$\cos \theta = \frac{8^2 + 4^2 - x^2}{2 \cdot 8 \cdot 4} = \frac{80 - x^2}{64}$$

And again, $\sin \theta$ is given by $\sqrt{1 - \cos^2 \theta}$. Hence, the cosine of $\phi$ is given:

$$\cos \phi = \cos(\gamma - \theta) = -\frac{59}{304} \cos \theta + \frac{77}{304} \sqrt{15} \sin \theta$$

Thus, for any $x$ we can calculate $y$, once again by the law of cosines:

$$y^2 = 4^2 + 19^2 - 2 \cdot 4 \cdot 19 \cdot \cos \phi = 377 - 152 \cos \phi$$

Calculating $y$ for $x = 5, 6, 7, 8, 9$ we find one integral value for $y$: $y = 17$ corresponding to $x = 6$.

So we have our answer:

$$\overline{AP} = 17$$
$$\overline{BP} = 6$$

**Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC**

We model the given triangle in the Cartesian plane by first placing $A$ at $(19, 0)$ and $C$ at $(0, 0)$. Then $B$ must lie on the circles with equations...
\[ x^2 + y^2 = 64 \quad \text{and} \quad (x - 19)^2 + y^2 = 484, \]

so we place \( B \) in the second quadrant at \( (d, e) \), where \( d = -59/38 \) and \( e = 77\sqrt{15}/38 \).

Next, we seek an interior point \( P = (x, y) \) such that \( x^2 + y^2 = 16, (x - 19)^2 + y^2 = m^2 \), and \( (x - d)^2 + (y - e)^2 = n^2 \) for positive integers \( m = \overline{AP} \) and \( n = \overline{BP} \). Since \( P \) is interior to triangle \( ABC \) and lies on the circle with equation \( x^2 + y^2 = 4 \), we have \( m \in \{16, 17, 18, 19, 20\} \) and \( n \in \{5, 6, 7, 8, 9\} \).

Solving the system
\[
\begin{cases}
  x^2 + y^2 = 16 \\
  (x - 19)^2 + y^2 = m^2
\end{cases}
\]
yields \( x = (377 - m^2)/38 \) and \( y = \sqrt{-m^4 + 754m^2 - 119025}/38 \). Substituting these values for \( x \) and \( y \) into \( (x - d)^2 + (y - e)^2 = n^2 \) for \( m \in \{16, 17, 18, 19, 20\} \), we find that only \( m = 17 \) produces a positive integer value for \( n \), namely \( n = 6 \). Hence \( P = (44/19, 16\sqrt{15}/19) \) with \( \overline{AP} = 17 \) and \( \overline{BP} = 6 \).

Comment by Albert Stadler, Herrliberg, Switzerland: There is no other interior point even if we get rid of the condition that \( \overline{CP} = 4 \). However, letting \( u = \overline{AP}, v = \overline{BP} \) and \( w = \overline{CP} \) and if we permit \( P \) to lie on a side of the triangle, then \((u, v, w) = (16, 6, 7)\) is the only additional point.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Vijaya Prasad Nalluri, Rajahmundry, India; Valentin Shopov, Munich, Germany; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Students at Taylor University, Upland, IN, ne Team 1: \{Hannah Peters, Ben Robison, Stevanni McCray\}
Team 2: \{Hannah King, Deborah Settles, Jackson Bronkema\}
Team 3: \{Gwyneth Terrett, Samantha Korn, Elissa Grace Moore\}, and the proposer.

- **5452**: Proposed by Roger Izard, Dallas, TX

Let point \( O \) be the orthocenter of a given triangle \( ABC \). In triangle \( ABC \) let the altitude from \( B \) intersect line segment \( AC \) at \( E \), and the altitude from \( C \) intersect line segment \( AB \) at \( D \). If \( AC \) and \( AB \) are unequal, derive a formula which gives the square of \( BC \) in terms of \( AC, AB, EO, \) and \( OD \).

**Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain**

Let \( a = BC, b = CA, c = AB, d = OD, e = EO, f = EA, \) and \( g = AD \). Applying the Pythagorean Theorem to \( \triangle ABE, \triangle BCE, \triangle OEA \) and \( \triangle OAD \), and using the fact that \( \triangle ABE \sim \triangle CAD \), because they are both right triangles with common angle at vertex \( A \), we obtain:

\[
\begin{align*}
  c^2 &= AB^2 + BE^2 + EA^2 = BC^2 - CE^2 + EA^2 = a^2 - (b - f)^2 + f^2 = a^2 - b^2 + 2bf, \\
  e^2 + f^2 &= EO^2 + EA^2 = OA^2 = OD^2 + AD^2 = d^2 + g^2, \quad \text{and}
\end{align*}
\]
\[
\frac{b}{g} = \frac{CA}{AD} = \frac{AB}{EA} = \frac{c}{f}.
\]

From these two last lines, we obtain
\[
e^2 + f^2 = d^2 + \frac{b^2 f^2}{c^2},
\]
\[
e^2 e^2 + e^2 f^2 = c^2 d^2 + b^2 f^2,
\]
and since \(b \neq c\) by hypothesis, we see that \(f^2 = \frac{e^2 (e^2 - d^2)}{b^2 - c^2}\), and from the equality \(c^2 = a^2 - b^2 + 2bf\) gives us \(a^2\) in terms \(b, c, e\) and \(d\). Namely,
\[
a^2 = (b^2 + e^2 - 2bf) = b^2 + c^2 - 2bc \sqrt{\frac{e^2 - d^2}{b^2 - c^2}}.
\]

Solution 2 by Kee-Wai Lau, Hong Kong, China

By the cosine formula, we have
\[
\frac{EO}{OA} = \sin \angle OAE = \cos \angle ACB = \frac{AC^2 + BC^2 - AB^2}{2(AC)(BC)}, \quad \text{and similarly}
\]
\[
\frac{OD}{OA} = \sin \angle OAD = \cos \angle ABC = \frac{AB^2 + BC^2 - AC^2}{2(AB)(BC)}.
\]
Hence,
\[
\frac{EO}{OD} = \frac{AB(AC^2 + BC^2 - AB^2)}{AC(AB^2 + BC^2 - AC^2)}.
\] (1)

Since \(AC \neq AB\), so \(\frac{EO}{OD} \neq \frac{AB}{AC}\) or \((AB)(OD) - (AC)(EO) \neq 0\). Solving (1) for \(BC^2\) we obtain
\[
BC^2 = \frac{(AB + AC)(AB - AC) ((AB)(OD) + (AC)(EO))}{(AB)(OD) - (AC)(EO)}.
\]

Also solved by Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Vijaya Prasad Nalluri, Rajahmundry, India; Albert Stadler, Herrliberg, Switzerland, and the proposer.

• 5453: Proposed by D.M. Bătinețu-Giurgiu, “Matei Basarab” National College, Bucharest, Romania and Neculai Stanciu, “George Emil Palade” School, Buzaǔ, Romania

If \(a, b, c \in (0, 1)\) or \(a, b, c \in (1, \infty)\) and \(m, n\) are positive real numbers, then prove that
\[
\frac{\log_a b + \log_b c}{m + n \log_a c} + \frac{\log_b c + \log_c a}{m + n \log_b a} + \frac{\log_c a + \log_a b}{m + n \log_c b} \geq \frac{6}{m + n}
\]

Solution 1 by Moti Levy, Rehovot, Israel
Let $x := \log_a b, \ y := \log_b c, \ z := \log_c a$. Then $xyz = \log_a b \log_b c \log_c a = 1$, and the condition $a, b, c \in (0, 1)$ or $a, b, c \in (1, \infty)$ implies that $x, y, z > 0$.

The original inequality may be rephrased as:

$$\frac{x + y}{m + z^{-1}n} + \frac{y + z}{m + x^{-1}n} + \frac{z + x}{m + y^{-1}n} \geq \frac{6}{m + n}, \quad xyz = 1, \ x, y, z > 0, \quad (1)$$

or as

$$\frac{3}{\sum_{cyc}(m + z^{-1}n)} \leq \frac{m + n}{2}. \quad (2)$$

Since the harmonic mean is less than or equal to the geometric mean,

$$\frac{3}{\sum_{cyc}(m + z^{-1}n)} \leq \sqrt[3]{\frac{m + z^{-1}n m + x^{-1}n m + y^{-1}n}{x + y y + z z + x}}.$$ 

Hence it is enough to prove (2):

$$\frac{m + z^{-1}n m + x^{-1}n m + y^{-1}n}{x + y y + z z + x} \leq \frac{(m + n)^3}{8},$$

$$\frac{1}{xyz} \frac{(n + mz)(n + mx)(n + my)}{(x + y)(x + z)(y + z)} \leq \frac{(m + n)^3}{8},$$

$$\frac{(n + mz)(n + mx)(n + my)}{(x + y)(x + z)(y + z)} \leq \frac{(m + n)^3}{8}. \quad (2)$$

Further simplification of (2) results in

$$\frac{n^3 + mn^2 x + mn^2 y + mn^2 z + m^2 nx y + m^2 n x z + m^2 n y z + m^3 x y z}{(x + y)(x + z)(y + z)} \leq \frac{(m + n)^3}{8},$$

$$\frac{n^3 + mn^2 (x + y + z) + m^2 n(x y + y z + x z) + m^3}{(x + y)(x + z)(y + z)} \leq \frac{(m + n)^3}{8} \quad (3)$$

Equating the left and right sides of (3) shows that the inequality (3) is equivalent to (4) and (5):

$$\frac{x + y + z}{(x + y)(x + z)(y + z)} \leq \frac{3}{8}, \quad (4)$$

$$\frac{xy + yz + xz}{(x + y)(x + z)(y + z)} \leq \frac{3}{8}, \quad (5)$$

We now use the $p, q, r$ notation:

$$p := x + y + z,$$

$$q := xy + yz + xz,$$

$$r := xyz.$$

In this notation, (4) and (5) become

$$\frac{p}{pq - r} \leq \frac{3}{8}, \quad (6)$$

$$\frac{q}{pq - r} \leq \frac{3}{8}. \quad (7)$$
In our case $r = 1$, which implies (by AM-GM inequality) that $p \geq 3$ and $q \geq 3$. Now proving (4) and (5) is straightforward:

\[
\frac{p}{pq - 1} \leq \frac{3}{8}, \\
3pq - 3 - 8p \geq 0, \\
3pq - 3 - 8p \geq p - 3 \geq 0.
\]

\[
\frac{q}{pq - 1} \leq \frac{3}{8}, \\
3pq - 3 - 8q \geq 0, \\
3pq - 3 - 8q \geq q - 3 \geq 0.
\]

**Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain**

Note that since $\log_a b = \frac{\ln b}{\ln a}$ and $a, b, c \in (0, 1)$ or $a, b, c \in (0, 1)$, all the logarithms in the proposed inequality are positive, so the right-hand side is positive.

We will apply the following parametrized Nesbitt’s inequality (see reference 1, theorem 7).

Let $x, y, z, tx + ky + lz, ty + kz + lx, tz + kx + ly$ be positive real numbers and let $-k - l < t < \frac{k + l}{2}$.

Then

\[
\frac{x}{tx + ky + lz} + \frac{y}{ty + kz + lx} + \frac{z}{tz + kx + ly} \geq 3t + k + l. 
\]

(1)

We will consider two inequalities, from which the stated problem will follow.

\[
\frac{\log_a b}{m + n \log_a c} + \frac{\log_a c}{m + n \log_b a} + \frac{\log_n a}{m + n \log_c b} \geq \frac{3}{m + n} 
\]

(2)

\[
\frac{\log_b c}{m + n \log_a c} + \frac{\log_c a}{m + n \log_b a} + \frac{\log_a b}{m + n \log_c b} \geq \frac{3}{m + n}. 
\]

(3)

Notice that the right-hand side of (2) is

\[
\text{RHS} = \frac{\ln b}{m \ln a + n \ln c} + \frac{\ln c}{m \ln b + n \ln a} + \frac{\ln a}{m \ln c + n \ln b} \geq \frac{3}{m + n}
\]

by the parametrized Nesbitt’s inequality with $t = 0$, $k = m$ and $l = n$, and $x = \ln b$, $y = \ln c$, and $z = \ln a$. It also should be noticed that in the last expression we may assume that all the $\ln$’s are positive.

Now, the right-hand side of (3) is

\[
\text{RHS} = \frac{\ln a \ln c}{m \ln a \ln b + n \ln b \ln c} + \frac{\ln a \ln b}{m \ln b \ln c + n \ln a \ln c} + \frac{\ln b \ln c}{m \ln a \ln c + n \ln a \ln b} \geq \frac{3}{m + n}
\]

by the parametrized Nesbitt’s inequality with $t = 0$, $k = m$ and $l = n$, and $x = \ln a \ln c$, $y = \ln a \ln b$, and $z = \ln b \ln c$. 

8
References:


Solution 3 by Soumitra Mandal, Chandar Nagore, India

\[ \sum_{cyc} \frac{\log_a b + \log_b c}{m + n \log_a c} = \sum_{cyc} \frac{\log b + \frac{\log a \log c}{\log b}}{m \log a + n \log c} \]
\[ = \sum_{cyc} \frac{\log b}{m \log a + n \log c} + \sum_{cyc} \frac{\log a \log c}{m \log a + n \log c} \]
\[ = \sum_{cyc} \frac{(\log b)^2}{n \log a \cdot \log b + n \log c \cdot \log b} + \sum_{cyc} \frac{m}{\log b \cdot \log c + \log b - \log a} \]
\[ \geq \frac{(\log a + \log b + \log c)^2}{(m + n)(\log a \cdot \log b + \log b - \log c + \log c \cdot \log a)} \]
\[ + \frac{\left( \frac{1}{\log a \cdot \log b} + \frac{1}{\log b \cdot \log c} + \frac{1}{\log c \cdot \log a} \right)^2}{(m + n)} \]
\[ \geq \frac{3}{m + n} + \frac{3}{m + n} = \frac{6}{m + n} \]

Editor’s Comments: Anna V. Tomova of Varna, Bulgaria approached the solution as follows: She showed that the left hand side of the inequality can be put into the canonical form of \( X + Y + \frac{1}{XY} \). She then showed that this canonical form has a global minimum at (1,1), forcing it to have a minimal value of 3, and working with this she produced the final result.

Bruno Salgueiro Fanego of Viveiro, Spain noted that the stated problem is a specific case of a more general result. Namely: If \( x, y, z \in (0, \infty) \) and \( xyz = 1 \), then
\[ \frac{x + y}{m + \frac{n}{z}} + \frac{y + z}{m + \frac{n}{x}} + \frac{z + x}{m + \frac{n}{y}} \geq \frac{6}{m + n} \]
He proved the more general result, and applied it to the specific case.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray of Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Shravan Sridhar, Udupi, India; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova of Varna, Bulgaria, and the proposer.

5454: Proposed by Arkady Alt, San Jose, CA

Prove that for integers \( k \) and \( l \), and for any \( \alpha, \beta \in (0, \frac{\pi}{2}) \), the following inequality holds:
\[ k^2 \tan \alpha + l^2 \tan \beta \geq \frac{2kl}{\sin(\alpha + \beta)} - (k^2 + l^2) \cot(\alpha + \beta) \]
Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the inequality by transposing

1) \( k^2 \left( \frac{\sin a + \cos(a + b)}{\cos a} \right) + t^2 \left( \frac{\sin b + \cos(a + b)}{\sin(a + b)} \right) \geq \frac{2kt}{\sin(a + b)} \)

Multiplying by \( \sin(a + b) \)

2) \( k^2 \left( \frac{\sin a \sin(a + b) + \cos a \cos(a + b)}{\cos a} \right) + t^2 \left( \frac{\sin b \sin(a + b) + \cos a \cos(a + b)}{\cos b} \right) \geq 2kt \)

3) \( k^2 \left( \frac{\cos b}{\cos a} \right) + t^2 \left( \frac{\cos a}{\cos b} \right) \geq 2kt \)

4) \( \frac{k^2 \cos^2 a + t^2 \cos^2 b}{\cos a \cos b} \geq 2kt \)

5) \( k^2 \cos^2 b + t^2 \cos^2 a \geq 2kt \cos a \cos b, \) and transposing,

6) \( k^2 \cos b - t \cos a \geq 0..\)

So we retrace our steps to obtain the original inequality.

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

First we consider the case when \( \alpha + \beta = \frac{\pi}{2}, \) then \( \sin(\alpha + \beta) = 1, \cot(\alpha + \beta) = 0, \) and \( \tan \beta = \cot \alpha. \) From these we have

\[
k^2 \tan \alpha + t^2 \tan \beta - \frac{2kt}{\sin(\alpha + \beta)} + \left( k^2 + t^2 \right) \cot(\alpha + \beta) = k^2 \tan \alpha + t^2 \cot \alpha - 2kl = \left( k \sqrt{\tan \alpha} - l \sqrt{\cot \alpha} \right)^2 \geq 0,
\]

which completes the proof when \( \alpha + \beta = \frac{\pi}{2}. \)

Now suppose that \( \alpha + \beta \neq \frac{\pi}{2} \) By using the identity \( \cot(\alpha + \beta) = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}, \) we have

\[
k^2 \tan \alpha + t^2 \tan \beta + \left( k^2 + t^2 \right) \cot(\alpha + \beta) - \frac{2kl}{\sin(\alpha + \beta)}
\]

\[
= k^2 \tan \alpha + t^2 \tan \beta + \left( k^2 + t^2 \right) \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} - \frac{2kl}{\sin(\alpha + \beta)}
\]

\[
= k^2 \tan^2 \alpha + k^2 \tan \alpha \tan \beta + t^2 \tan \beta \tan \alpha + t^2 \tan^2 \beta + \left( k^2 + t^2 \right) \frac{1}{\tan \alpha + \tan \beta} - \frac{2kl}{\sin(\alpha + \beta)}
\]

\[
= k^2 \tan^2 \alpha + t^2 \tan^2 \beta + \left( k^2 + t^2 \right) \frac{1}{\tan \alpha + \tan \beta} - \frac{2kl}{\sin(\alpha + \beta)}
\]

\[
= \frac{k^2 (1 + \tan^2 \alpha) + t^2 (1 + \tan^2 \beta)}{\sin \alpha + \sin \beta} - \frac{2kl}{\sin(\alpha + \beta)}
\]
\[ \frac{k^2 \sec^2 \alpha + l^2 \sec^2 \beta}{\cos \alpha \cos \beta} - \frac{2kl}{\sin(\alpha + \beta)} = \frac{k^2 \cos \beta}{\cos \alpha} + \frac{l^2 \cos \alpha}{\cos \beta} - \frac{2kl}{\sin(\alpha + \beta)} = \left( \sqrt{\frac{\cos \beta}{\cos \alpha} - l \sqrt{\frac{\cos \alpha}{\cos \beta}}} \right)^2 \geq 0. \]

\textit{Editor’s Note:} Most of the solvers mentioned that the inequality holds for all real values of \( k \) and \( l \). David Stone and John Hawkins of Georgia Southern University when a bit further. They stated: “the conditions that \( \alpha \) and \( \beta \) be first quadrant angles is an easy way to make \( \sin(\alpha + \beta) \neq 0 \) and \( \tan \alpha, \tan \beta, \cot(\alpha + \beta) \) be defined and guarantee that \( \cos \alpha \cos \beta \sin(\alpha + \beta) > 0. \)” But the proof shows that the inequality would be true for any values of \( \alpha \) and \( \beta \) which satisfy these conditions.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Daniel Sitaru, “Theodor Costescu” National Economic College, Severin Mehedinti; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5455: Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Find all real solutions to the following system of equations:

\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc} \]

\[ a + b + c = abc + \frac{8}{27} (a + b + c)^3 \]

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Suppose \( a, b, c \) are real numbers satisfying our system. Consider the polynomial

\[ g(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc. \]

The first equation of our original system implies \( ab + ac + bc = 1 \). So

\[ g(x) = x^3 - \lambda x^2 + x - \mu \]

where \( \lambda = a + b + c \) and \( \mu = abc \). Note that the second equation of our original system can be written as \( \lambda = \mu + \frac{8}{27} \lambda^3 \). We make the usual substitution to get a depressed cubic:
\( g(x + \lambda/3) = x^3 + px + q \) where

\[ p = 1 - \frac{1}{3} \lambda^2 \ \text{and} \ q = -\frac{2}{27} \lambda^3 + \frac{1}{3} \lambda - \mu. \]

Using \( \lambda = \mu + \frac{8}{27} \lambda^3 \) we have

\[ q = \frac{2}{9} \lambda^3 - \frac{2}{3} \lambda \]

which we factor to get

\[ q = -\frac{2}{3} \lambda \left( 1 - \frac{1}{3} \lambda^2 \right) = -\frac{2}{3} \lambda p. \]

The discriminant of \( g(x + \lambda/3) \) is

\[
D = -4p^3 - 27q^2
\]

\[
= -4p^3 - 12\lambda^2 p^2
\]

\[
= -4p^2(\lambda + 3\lambda^2)
\]

\[
= -4p^2 \left( 1 + \frac{8}{3} \lambda^2 \right)
\]

Note that \( D \geq 0 \) if and only if \( p = 0 \). Recall that a real cubic polynomial has three real roots if and only if its discriminant is \( \geq 0 \). Thus \( g(x + \lambda/3) \) has three real roots if and only if \( p = 0 \) if and only if \( \lambda = \pm \sqrt{3} \). Note that when \( \lambda = \pm \sqrt{3} \) we have \( g(x + \lambda/3) = x^3 \), and hence \( g(x) = (x - \lambda/3)^3 \). Therefore the only solutions to the original system are

\[ a = b = c = \frac{\sqrt{3}}{3} \quad \text{and} \quad a = b = c = -\frac{\sqrt{3}}{3}. \]

**Solution 2 by Moti Levy, Rehovot, Israel**

\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc} \] implies \( ab + bc + ca = 1 \).

Substitution of \( abc = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \) in the second equation gives

\[
a + b + c - \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} - \frac{8}{27} (a + b + c)^3 = 0,
\]

\[
\frac{(a + b)(a + c)(b + c)}{abc + ac + bc} - \frac{8}{27} (a + b + c)^3 = 0,
\]

\[
(a + b)(a + c)(b + c) - \frac{8}{27} (a + b + c)^3 = 0. \quad (1)
\]

Let \( x = a + b \), \( y = b + c \) and \( z = c + a \) then (1) becomes

\[ xyz - \left( \frac{x + y + z}{3} \right)^3 = 0, \]

or \( \sqrt[3]{xyz} = \frac{x + y + z}{3} \). The geometric mean is equal to the arithmetic mean if and only if \( x = y = z \) which implies that \( a = b = c \).

Therefore the system of equation has only two solutions:

\[ a = b = c = \frac{1}{\sqrt{3}}, \quad a = b = c = -\frac{1}{\sqrt{3}} \]
Solution 3 by Kee-Wai Lau, Hong Kong, China

Let $p = a + b + c$, $q = ab + bc + ca$, and $r = abc$. The first given equation becomes

$$q = 1$$

and the second equations becomes

$$r = p - \frac{8p^3}{27}.\tag{2}$$

It can be checked readily that

$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 = (a - b)^2(b - c)^2(c - a)^2.\tag{3}$$

Using (1) and (2) we reduce the left side of (3) to

$$-4(\frac{p^2}{3})^2(8\frac{p^3}{27})^2,$$

which is non-positive. Since the right side of (3) is nonnegative, so both sides of (3) equal to zero. It follows that $p^2 = 3$ and by (2), $r = \frac{p}{9}$. Moreover, either $a = b$ or $b = c$ or $c = a$. By symmetry we only consider the case $a = b$. Hence either $2a + c = \sqrt{3}$, $a^2 + 2ac = 1$, or $2a + c = -\sqrt{3}$, $a^2 + 2ac = 1$, giving respectively $a = c = \frac{1}{\sqrt{3}}$ and $a = c = -\frac{1}{\sqrt{3}}$. Thus the solutions to the original system are precisely $(a, b, c) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right).

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Le Van, Ho Chi Minh City, Vietnam; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5456: Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

Let $k$ be a positive integer. Calculate

$$\lim_{x \to \infty} e^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left( e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right).$$

Solution 1 Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

By the Taylor formula we have

$$e^x = 1 + x + x^2/2! + \cdots + x^n/n! + \int_0^x \frac{(x-t)^n}{n!} e^t dt.$$ 

It follows that

$$e^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left( e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right)$$

$$= \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \int_0^x \frac{(x-t)^n}{n!} e^{-t} dt$$

$$= \frac{1}{k!} \sum_{n=k}^{\infty} (-1)^n \int_0^x \frac{t^n}{(n-k)!} e^{-t} dt$$

$$= \frac{1}{k!} \sum_{n=0}^{\infty} (-1)^{n+k} \int_0^x \frac{t^{n+k}}{n!} e^{-t} dt$$

$$= \frac{(-1)^k}{k!} \int_0^x t^k e^{-2t} dt.$$
The limit as \( x \to +\infty \) is given by
\[
\frac{(-1)^k}{k!} \int_0^\infty t^k e^{-2t} dt = \frac{(-1)^k}{2^{k+1}}.
\]

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

\[
\lim_{x \to +\infty} e^{-x} \sum_{n=0}^\infty (-1)^n \binom{n}{k} \left( e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right)
\]

\[
= \lim_{x \to +\infty} e^{-x} \sum_{n=0}^\infty (-1)^n \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} \sum_{p=n+1}^\infty \frac{x^p}{p!}
\]

\[
= \frac{1}{k!} \lim_{x \to +\infty} e^{-x} \frac{d^k}{dt^k} \left( \sum_{n=0}^\infty \frac{x^n}{n!} n \sum_{p=0}^n (-1)^p t^p \right) \bigg|_{t=1}
\]

\[
= \frac{1}{k!} \lim_{x \to +\infty} e^{-x} \frac{d^k}{dt^k} \left( \frac{1}{1+t} \left( \sum_{n=0}^\infty \frac{x^n}{n!} - \sum_{n=0}^\infty \frac{(-xt)^n}{n!} \right) \right) \bigg|_{t=1}
\]

\[
= \frac{1}{k!} \lim_{x \to +\infty} \frac{d^k}{dt^k} \left( \frac{1}{1+t} \lim_{x \to +\infty} (1 - e^{-x(1+t)}) \right) \bigg|_{t=1}
\]

\[
= \frac{1}{k!} \left( \frac{d^k}{dt^k} \frac{1}{1+t} \right) \bigg|_{t=1} = \frac{(-1)^k}{(1+t)^{k+1}} = \frac{(-1)^k}{2^{k+1}}.
\]

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Repeated integration by parts yields
\[
\int_0^x e^{-t} \frac{t^n}{n!} dt = - \frac{e^{-x} x^n}{n!} + \int_0^x d-t \frac{t^{n-1}}{(n-1)!} dt = - e^{-x} \left( \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x \right) + \int_0^x e^{-t} dt
\]

\[
= - e^{-x} \left( \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 \right) + 1.
\]

So,
\[
e^{-x} - \left( \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 \right) = e^x \int_0^x e^{-t} \frac{t^n}{n!} dt
\]

and
\[
e^{-x} \sum_{n=0}^\infty (-1)^n \binom{n}{k} \left( e^x - \left( \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 \right) \right) = \frac{1}{k!} \sum_{n=k}^\infty \frac{(-1)^n}{(n-k)!} \left( \int_0^x e^{-t} \frac{t^n}{n!} dt \right)
\]
\[
\frac{(-1)^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \int_0^x e^{-t} t^n dt \right)
\]

\[
= \frac{(-1)^k}{k!} \int_0^x e^{-2t} t^k dt \to \frac{(-1)^k}{k!2^{k+1}} \int_0^\infty e^{-2t} t^k dt = \frac{(-1)^k}{2^{k+1}}, \text{ as } x \to \infty.
\]

**Solution 4 by Kee-Wai Lau, Hong Kong, China**

We show that the given limit equals \((-1)^k2^{-(k+1)}\).

For real \(x\) let

\[
f(x) = e^{-x}x^k = \sum_{m=k}^{\infty} (-1)^{m-k} \frac{x^m}{m-k}!
\]

so that

\[
f^n(0) = \begin{cases} 
0, & 0 \leq n \leq k - 1 \\
(-1)^{n-k}n(n-1)\cdots(n-k+1), & n \geq k
\end{cases}
\]

where \(f^n(x)\) is the \(n\)th derivative of \(f(x)\).

According to problem 3.89(a) on pp124, 227 of the book [Ovidiu Furdui; Limits, Series, and Fractional Part Integrals, Springer 2013] we have

\[
\sum_{n=0}^{\infty} f^n(0) \left( e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t} f(t) dt.
\]

Hence,

\[
e^{-x} \sum_{n=k}^{\infty} (-1)^n \binom{n}{k} \left( e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) = \frac{(-1)^k}{k!} \int_0^x e^{-2t} t^k dt
\]

\[
= \frac{(-1)^k}{2^{k+1}k!} \int_0^{2x} e^{-t} t^k dt.
\]

Now our result for the limit follows from the well-known fact that \(\int_0^\infty e^{-t} t^k dt = k!\).

**Also solved by Moti Levy, Rehovot, Israel; Anna V. Tomova, Varna, Bulgaria, and the proposer.**

*Editor’s Comment:* In Anna’s solution to 5456 she acknowledged contributing conversations with Peter Breuer and Joachim Domsta of Bulgaria.